1. Consider the region $R$ in $\mathbb{R}^2$ shown below at right. In this problem, you will do a change of coordinates to evaluate:

$$\int \int_R x - 2y \, dA$$

(a) Find a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which takes the unit square $S$ to $R$. Write your answer both as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and as $T(u, v) = (au + bv, cu + dv)$, and check your answer with the instructor.

(b) Compute $\int \int_R x - 2y \, dA$ by relating it to an integral over $S$ and evaluating that. Check your answer with the instructor.

2. Another simple type of transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a translation, which has the general form $T(u, v) = (u + a, v + b)$ for a fixed $a$ and $b$.

(a) If $T$ is a translation, what is its Jacobian matrix? How does it distort area?

(b) Consider the region $S = \{ u^2 + v^2 \leq 1 \}$ in $\mathbb{R}^2$ with coordinates $(u, v)$, and the region $R = \{ (x - 2)^2 + (y - 1)^2 \leq 1 \}$ in $\mathbb{R}^2$ with coordinates $(x, y)$.

Make separate sketches of $S$ and $R$.

(c) Find a translation $T$ where $T(S) = R$.

(d) Use $T$ to reduce

$$\int \int_R x \, dA$$

to an integral over $S$, and then evaluate that new integral using polar coordinates.

(e) Check your answer in (d) with the instructor.

Problems 3 and 4 on the back.
3. Consider the region \( R \) shown below. Here the curved left side is given by \( x = y - y^2 \). In this problem, you will find a transformation \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which takes the unit square \( S = [0, 1] \times [0, 1] \) to \( R \).

(a) As a warm up, find a transformation that takes \( S \) to the rectangle \([0, 2] \times [0, 1]\) which contains \( R \).

(b) Returning to the problem of finding \( T \) taking \( S \) to \( R \), come up with formulas for \( T(u, 0), T(u, 1), T(0, v), \) and \( T(1, v) \). Hint: For three of these, use your answer in part (a).

(c) Now extend your answer in (b) to the needed transformation \( T \). Hint: Try “filling in” between \( T(0, v) \) and \( T(1, v) \) with a straight line.

(d) Compute the area of \( R \) in two ways, once using \( T \) to change coordinates and once directly.

4. If you get this far, evaluate the integrals in Problems 1 and 2 directly, without doing a change of coordinates. It’s a fun-filled task...