In these notes, we discuss Gauss’s law and why it is interesting not only for physics, but also from a mathematical viewpoint.

1 Statement

The statement of Gauss’s law is as follows. The (net) charge enclosed by a closed surface $S$ is

$$Q = \epsilon_0 \int \int_S \vec{E} \cdot \vec{n} dS$$

where $\vec{E}$ is the electric field and $\epsilon_0$ is a constant, called the permittivity of free space. More details can be found in the textbook, § 16.7 after Example 5 and § 16.9 after Example 2.

2 Sketch of proof

Gauss’s law follows from Coulomb’s law and the divergence theorem.

By Coulomb’s law, an electric charge $Q$ at the origin produces the electric field

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi \epsilon_0} \frac{\vec{r}}{|\vec{r}|^3}$$

where $\vec{r} = (x, y, z)$ is the position vector. Such a vector field is sometimes called an inverse square field, because its magnitude

$$|\vec{E}(\vec{r})| = \frac{Q}{4\pi \epsilon_0} \frac{|\vec{r}|}{|\vec{r}|^3} = \frac{Q}{4\pi \epsilon_0} \frac{1}{|\vec{r}|^2}$$

is proportional to the inverse of the square of the distance to the origin (or some other base-point). In symbols: $|\vec{E}(\vec{r})| \propto \frac{1}{|\vec{r}|^2}$.

Step 1: Sphere around the origin

Let $S$ be the sphere of radius $R$ centered at the origin, defined by the equation $x^2 + y^2 + z^2 = R^2$. Orient $S$ outward, so that the normal vector $\vec{n}$ points away from the origin. The flux of $\vec{E}$ across
\[
S \text{ is } \int_S \vec{E} \cdot \vec{n} \, dS = \int_S (\vec{E}) \cdot dS \text{ because } \vec{E} \text{ is parallel to } \vec{n}
\]

\[
= \int_S \frac{Q}{4\pi \varepsilon_0 |\vec{r}|^2} \, dS
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \int_S \frac{1}{R^2} \, dS
\]

\[
= \frac{Q}{4\pi \varepsilon_0 R^2} \int_S 1 \, dS
\]

\[
= \frac{Q}{4\pi \varepsilon_0 R^2} \text{Area}(S)
\]

\[
= \frac{Q}{4\pi \varepsilon_0 R^2} (4\pi R^2)
\]

\[
= \frac{Q}{\varepsilon_0}.
\]

**Step 2: Weird surface around the origin**

Now let \( S' \) be some arbitrary closed surface enclosing the origin. Orient \( S' \) outward. What is the flux of \( \vec{E} \) across \( S' \)? We can find the answer using the divergence theorem.

Writing \( \rho = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \) consider the vector field \( \vec{F} = \frac{1}{\rho^2} \vec{r} = \frac{1}{\rho^3}(x, y, z) \), which is just \( \vec{E} \) scaled by a constant. Noting \( \frac{\partial \rho}{\partial x} = \frac{x}{\rho} \), the divergence is

\[
\text{div} \, \vec{F} = \frac{\partial}{\partial x}(x \rho^{-3}) + \frac{\partial}{\partial y}(y \rho^{-3}) + \frac{\partial}{\partial z}(z \rho^{-3})
\]

\[
= (1)\rho^{-3} + x(-3\rho^{-4})\left(\frac{\rho}{x}\right) + (1)\rho^{-3} + y(-3\rho^{-4})\left(\frac{\rho}{y}\right) + (1)\rho^{-3} + z(-3\rho^{-4})\left(\frac{\rho}{z}\right)
\]

\[
= 3\rho^{-3} - 3\rho^{-5}(x^2 + y^2 + z^2)
\]

\[
= 3\rho^{-3} - 3\rho^{-5}(\rho^2)
\]

\[
= 3\rho^{-3} - 3\rho^{-3}
\]

\[
= 0.
\]

Therefore we have

\[
\text{div} \, \vec{E} = \text{div} \left( \frac{Q}{4\pi \varepsilon_0} \vec{F} \right)
\]

\[
= \frac{Q}{4\pi \varepsilon_0} \text{div} \, \vec{F}
\]

\[
= 0.
\]

or in words, \( \vec{E} \) is incompressible.
By the divergence theorem, the flux across $S'$ is

$$\int\int_{S'} \vec{E} \cdot \vec{n} dS = \int\int_{S} \vec{E} \cdot \vec{n} dS = \frac{Q}{\epsilon_0}$$

where $S$ is a sphere around the origin (of any radius).

Let us describe the argument in more detail. Pick a giant sphere $S$ which encompasses all of $S'$, and orient $S$ outward. Let $D$ be the solid region between $S$ and $S'$, and orient the boundary of $D$ so that the normal vector points out of $D$, yielding $\partial D = S - S'$. Applying the divergence theorem to the region $D$, we obtain

$$\int\int_{\partial D} \vec{E} \cdot \vec{n} dS = \int\int\int_{D} \text{div} \vec{E} dV = \int\int\int_{D} 0 dV = 0$$

which can be interpreted as

$$0 = \int\int_{\partial D} \vec{E} \cdot \vec{n} dS = \int\int_{S - S'} \vec{E} \cdot \vec{n} dS = \int\int_{S} \vec{E} \cdot \vec{n} dS - \int\int_{S'} \vec{E} \cdot \vec{n} dS.$$

In other words, the flux across $S$ is the same as the flux across $S'$, as claimed above.

Note that the divergence theorem does not apply to the region enclosed by $S$, i.e. the punctured solid ball defined by $0 < x^2 + y^2 + z^2 \leq R^2$, because that region is not closed. The singularity at the origin prevents us from using the divergence theorem.

**Step 3: Weird surface not around the origin**

Now what if $S''$ is a closed surface that does not enclose the origin? Then $S''$ is the boundary of a solid region $D''$ which does not contain the origin, and the divergence theorem applies:

$$\int\int_{S''} \vec{E} \cdot \vec{n} dS = \int\int_{\partial D''} \vec{E} \cdot \vec{n} dS = \int\int\int_{D''} \text{div} \vec{E} dV = \int\int\int_{D''} 0 dV = 0.$$
In short, we have shown that if $S$ is a closed surface (with outward orientation), then the flux of the electric field $\vec{E}$ across $S$ is

$$\int \int_S \vec{E} \cdot \vec{n} \, dS = \begin{cases} \frac{Q}{\epsilon_0} & \text{if } S \text{ encloses the origin} \\ 0 & \text{if } S \text{ does not enclose the origin.} \end{cases}$$

We can rewrite this as

$$\epsilon_0 \int \int_S \vec{E} \cdot \vec{n} \, dS = \begin{cases} Q & \text{if } S \text{ encloses the origin} \\ 0 & \text{if } S \text{ does not enclose the origin} \end{cases}$$

which is equal to the (net) charge enclosed by $S$. This proves Gauss’s law in the case of a single (pointlike) charge.

**Step 4: Many electric charges**

For a finite system of charges $Q_i$ at positions $\vec{r}_i$, consider the electric field $\vec{E}_i(\vec{r}) = \frac{Q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|^3}$ produced by each charge. The total electric field $\vec{E}$ is their superposition:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_N.$$

For any closed surface $S$ (oriented outward), $\epsilon_0$ times the flux of the electric field $\vec{E}$ across $S$ is

$$\epsilon_0 \int \int_S \vec{E} \cdot \vec{n} \, dS = \epsilon_0 \int \int_S \left( \sum_{i=1}^N \vec{E}_i \right) \cdot \vec{n} \, dS$$

$$= \sum_{i=1}^N \epsilon_0 \int \int_S \vec{E}_i \cdot \vec{n} \, dS$$

$$= \sum_{i \text{ such that } S \text{ encloses the position } \vec{r}_i} Q_i$$

$$= \text{net charge enclosed by } S.$$

This proves Gauss’s law in the case of finitely many (pointlike) charges.

A similar argument proves Gauss’s law in the case of a continuous distribution of electric charges, described by a charge density function.

3  Again, which vector fields are curls?

In section § 16.8, we asked the question: How do we know if a vector field $\vec{F}$ is the curl of some vector field $\vec{G}$? We found a necessary condition: a curl is always incompressible, i.e.

$$\text{div(curl } \vec{G} \text{)} \equiv 0.$$  

Then we wondered if that condition is sufficient: Given div $\vec{F} = 0$, can we conclude that $\vec{F}$ is the curl of some vector field? We provided the answer – NO! – without justification. Now we can justify that negative answer.
Consider the inverse square vector field \( \vec{F} = \frac{1}{\rho^3} \vec{r} = \frac{1}{\rho^3}(x, y, z) \). It is incompressible, i.e.

\[
\text{div } \vec{F} = 0
\]
as computed in (1). However, \( \vec{F} \) is not the curl of a vector field. Indeed, we have found a closed surface \( S \) (say, a sphere centered at the origin) such that the flux of \( \vec{F} \) across \( S \) is non-zero:

\[
\int \int_S \vec{F} \cdot \vec{n} \, dS = 4\pi \neq 0.
\]

Therefore \( \vec{F} \) is not a curl.

Here we used Stokes’ theorem, which implies that the flux of curl \( \vec{G} \) across any closed surface must be zero:

\[
\int \int_S \text{curl } \vec{G} \cdot \vec{n} \, dS = \int_{\partial S} \vec{G} \cdot d\vec{r} = \int_{\emptyset} \vec{G} \cdot d\vec{r} = 0.
\]  
(2)

**Remark 3.1.** With a bit of topology, one can show that property (2) characterizes curls: A vector field is a curl if and only if its flux across any closed surface is zero.

Recall that there is a partial converse. The condition of being incompressible is sometimes sufficient for being a curl.

**Proposition 3.2.** Let \( \vec{F} \) be a continuously differentiable vector field on all of \( \mathbb{R}^3 \). If \( \vec{F} \) satisfies \( \text{div } \vec{F} \equiv 0 \), then \( \vec{F} \) is the curl of some vector field. In words: a vector field on all of \( \mathbb{R}^3 \) is a curl if and only if it is incompressible.

**Proof.** The key point is that in \( \mathbb{R}^3 \), a closed surface \( S \) always bounds a solid region \( D \).

Let \( \vec{F} \) be a vector field satisfying \( \text{div } \vec{F} \equiv 0 \) and let \( S \) be any closed surface. Let \( D \) be the solid region bounded by \( S \). Then the flux of \( \vec{F} \) across \( S \) (oriented outward) is

\[
\int \int_S \vec{F} \cdot \vec{n} \, dS = \int \int_{\partial D} \vec{F} \cdot \vec{n} \, dS = \int \int_D \text{div } \vec{F} \, dV = \int \int_D 0 \, dV = 0.
\]

By 3.1, \( \vec{F} \) is the curl of some vector field. \( \square \)