WARING'S PROBLEM WITH POLYNOMIAL SUMMANDS

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1. Introduction

Let $f(x)$ be an integer valued polynomial with no fixed integer divisor $\geqslant 2$, i.e. for no integer $d \geqslant 2$ does $d | f(x)$ for all integers x. One generalization of the famous Waring problem is to determine whether for large enough s, the equation

$$f(x_1) + f(x_2) + \cdots + f(x_s) = n \tag{1.1}$$

is solvable in positive integers $x_1, \ldots, x_s$ for sufficiently large integers n. The existence of such s for every f was established by Kamke [5] in 1921. Subsequent authors (Pillai, Hua ([2],[3],[4]), Vinogradov, Načaev [7] and others) have studied the problem of bounding $G(f)$, the least s for which (1.1) is solvable for all large n.

Questions of local solubility of (1.1), that is solubility of the congruence

$$f(x_1) + f(x_2) + \cdots + f(x_s) \equiv n \pmod{q}, \tag{1.2}$$

play a more important and complicated role in this problem than in the classical Waring problem. Let $\Gamma_0(f)$ denote the least number s so that (1.2) is solvable for every pair n, q. It is well known that $\Gamma_0(x^k) \leqslant 4k$ for every k, but Hua [4] found that for every k, the polynomial

$$f_k(x) = \sum_{j=1}^{k} (-1)^{k-j} 2^{j-1} \binom{x}{j}$$

satisfies $\Gamma_0(f_k) \geqslant 2^k - 1$ (take $s = 2^k - 2$, $q = 2^k$ and $n = (-1)^k$ in (1.2)). Clearly $G(f) \geqslant \Gamma_0(f)$, but one can say more by restricting the values of n under consideration, as has been done by several authors in the case $f(x) = x^4$ (e.g. [1], [6]).

The singular series

$$\mathfrak{S}_{s,f}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{1}{q} \sum_{r=1}^{q} e(af(r)/q) \right)^s e(-an/q),$$

where $e(z) = e^{2\pi i z}$, encapsulates the local solubility information. In particular, $\mathfrak{S}_{s,f}(n) \geqslant 0$ for every n and $\mathfrak{S}_{s,f}(n) > 0$ if and only if (1.2) is soluble for every q.

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Define $\overline{G}(f)$ to be the least number s so that for every $\delta > 0$ and every $n > n_0(\delta)$ with $\mathfrak{S}_{s,f}(n) \geqslant \delta$, (1.1) is soluble. The reason for taking $\mathfrak{S}_{s,f}(n) \geqslant \delta$ instead of $\mathfrak{S}_{s,f}(n) > 0$ is that we wish to exclude from consideration certain n lying in sparse sequences for which (1.1) is insoluble but $\mathfrak{S}_{s,f}(n) > 0$. For example, taking $f(x) = x^4$, $s = 15$ and $n_j = 79 \cdot 16^j$ $(j = 0, 1, \dots)$, it can be shown that (1.1) is not soluble for $n = n_j$, that $\mathfrak{S}_{s,f}(n_j) > 0$ for all j, and that $\mathfrak{S}_{s,f}(n_j) \to 0$ as $j \to \infty$. It is known that $G(x^4) = 16$ (see [1]) and that $\overline{G}(x^4) \leqslant 11$ (see [6]).

It has been known for several decades that a standard diminishing ranges argument combined with Vinogradov's bounds for exponential sums yields

$$\overline{G}(f) \leqslant k(4 \log k + O(\log \log k)) \tag{1.3}$$

uniformly for all f of degree k. In the classical Waring problem, Wooley [8] has shown that

$$G(x^k) \leqslant k(\log k + O(\log \log k)).$$

The new methods which have been successful in reducing the upper bounds in the classical case, however, do not apply directly in the more general case. The only improvement to (1.3) has been by Wooley [10], who has reduced the upper bound for $\overline{G}(f)$ for certain lacunary polynomials f (polynomials with few non-zero coefficients). For the precise statement, let

$$f(x) = \sum_{i=1}^{t} a_i x^{k_i}, \quad k = k_1 > k_2 > \cdots > k_t, \tag{1.4}$$

where the numbers a_i are integers, $a_1 > 0$ and $k \geqslant 2$. We do not lose any generality in assuming that the a_i are integers, for (1.1) is soluble for a given function $f(x) = g(x)$ and given $n = m$ if and only if the corresponding equation with $f(x) = dg(x)$ is soluble with $n = dm$.

Theorem [10, Theorem 9]. *We have*

(i) $\overline{G}(f) \leqslant 2k(\log k + \log t + \log \log k + O(1))$,

(ii) $\overline{G}(f) \leqslant (1 + o(1))k \log k$ *as* $k \to \infty$ *provided either*

(a) $t = o(\sqrt{\log k})$ *and* $\log(k_2 k_3 \cdots k_t) = o(\log k)$; *or*

(b) $k_2 = o\left(\dfrac{\log k}{\log \log k}\right)$.

This gives an upper bound for $\overline{G}(f)$ of the same strength as in the classical setting when both t and k_2 are very small. The purpose of this paper is to show that $\overline{G}(f) \leqslant (1 + o(1))k \log k$ for a wider class of polynomials, in particular those where $k - k_t$ is small.

Theorem 1. *With f defined by (1.4), we have*

$$\overline{G}(f) \leqslant k(\log k + 2 \log t + \log(k - k_t) + \log \log k + O(1)).$$

The main idea is to modify the efficient differencing method of [8] to handle general polynomial summands, and use these bounds together with the exponential sum bounds in [10] and the diminishing ranges method.

2. A mean value theorem

Let

$$f(x) = \sum_{i=1}^{t} a_i x^{k_i},$$

where

$$k = k_1 > k_2 > \cdots > k_t = k_1 - r \geqslant 1$$

and each a_i is a non-zero integer. We also assume that $a_1 > 0$, $k \geqslant 4$, $r \leqslant k/2$ and $t \geqslant 2$. With f fixed, define for positive integers w

$$g(x; w) = w^{-k_t} f(xw) = \sum_{i=1}^{t} w^{k_i - k_t} a_i x^{k_i}.$$

Let $\mathscr{A}(P, R)$ denote the set of positive integers $n \leqslant P$ which have no prime factors exceeding R. Let $I_s(P, R; w)$ denote the number of solutions of

$$\sum_{i=1}^{s} (g(x_i; w) - g(y_i; w)) = 0, \qquad x_i, y_i \in \mathscr{A}(P, R) \quad (1 \leqslant i \leqslant s).$$

We say a number λ_s is admissible if for every $\varepsilon > 0$, there is a positive $\eta = \eta(\varepsilon, s)$ such that if $R \leqslant P^\eta$ then

$$I_s(P, R; w) \ll P^{\lambda_s + \varepsilon} \qquad \text{uniformly in } w \geqslant 1.$$

The implied constant may depend on ε, s and f only.

In the following, vectors are denoted by boldface symbols with subscripts indicating their dimension, e.g. $\boldsymbol{m}_j$ stands for $(m_1, m_2, \ldots, m_j)$. If $\Psi(z) = \Psi(z; \boldsymbol{h}_j, \boldsymbol{m}_j, w)$ is a polynomial in $2j + 2$ variables with non-negative coefficients, let

$$S_s(P, Q, R) = S_s(P, Q, R; \Psi; \boldsymbol{m}_j; w)$$

be the number of solutions of

$$\Psi(z; \boldsymbol{h}_j, \boldsymbol{m}_j, w) + \sum_{i=1}^{s} g(x_i; w m_1 \cdots m_j) = \Psi(z'; \boldsymbol{h}'_j, \boldsymbol{m}_j, w) + \sum_{i=1}^{s} g(y_i; w m_1 \cdots m_j) \tag{2.1}$$

with

$$P/2 \leqslant z, z' \leqslant P; \qquad \forall i: \quad 1 \leqslant h_i, h'_i \leqslant P/m_i^{k_t};\ x_i, y_i \in \mathscr{A}(Q, R).$$

Also, let

$$T_s(P, Q, R, M) = T_s(P, Q, R, M; \Psi; \boldsymbol{m}_{j+1}; w)$$

denote the number of solutions of

$$\Psi(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w)-\Psi(z';\boldsymbol{h}_j,\boldsymbol{m}_j,w) + m_{j+1}^{k_t}\sum_{i=1}^{s}(g(x_i;wm_1\cdots m_{j+1})-g(y_i;wm_1\cdots m_{j+1}))=0 \tag{2.2}$$

with

$$P/2\leqslant z,z'\leqslant P; z\equiv z' \,(\mathrm{mod}\ m_{j+1}^{k_t});\quad \forall i:\ 1\leqslant h_i\leqslant P/m_i^{k_t};\ x_i,y_i\in\mathscr{A}(Q/M,R).$$

From now on it will be assumed that $\Psi(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w)$, interpreted as a polynomial in z with coefficients depending on $\boldsymbol{h}_j,\boldsymbol{m}_j,w$, has degree between 2 and k and the leading coefficient is positive whenever the variables h_i,m_i,w are all positive. The notation $o(1)$ will mean a function which tends to 0 as $P\to\infty$, the rate of which depends only on f and s. We also write $\Psi'(z)$ or $\Psi'(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w)$ for the partial derivative of Ψ with respect to z, and assume it is non-zero whenever $P/2\leqslant z\leqslant P$ and each variable h_i,m_i,w is positive. We shall also suppose that $R=P^\eta$ for some fixed $\eta>0$ depending on f,s,ε. Implied constants depend only on f,s,η,ε unless otherwise specified.

Our first lemma is a direct generalization of Lemma 2.2 of [8].

Lemma 2.1. *If* $1<M<Q<P$ *then*

$$S_s(P,Q,R;\Psi,\boldsymbol{m}_j;w)\ll \left(\prod_{i=1}^{j}P/m_i^{k_t}\right)(MR)^{2s}P^{o(1)}\max_{M\leqslant m_{j+1}\leqslant MR}T_s(P,Q,R,M;\Psi;\boldsymbol{m}_{j+1};w).$$

Proof. As in [8], the notation $s_0(n)$ stands for the square-free kernel of n, i.e. $s_0(n)=\prod_{p|d}p$, and $x\mathscr{D}(M)y$ means some $d|x, d\leqslant M$ satisfies $s_0(x/d)|y$.

We partition the solutions $z,z',\boldsymbol{h}_j,\boldsymbol{h}_j',\boldsymbol{x}_s,\boldsymbol{y}_s$ of (2.1) into three classes: S_1 is the number of solutions with some $x_i\leqslant M$ or some $y_i\leqslant M$, S_2 is the number of remaining solutions with $x_i\mathscr{D}(M)\Psi'(z)$ or $y_i\mathscr{D}(M)\Psi'(z')$ for some i, and S_3 is the number of solutions not counted in S_1 or S_2. Clearly $S_s(P,Q,R)\leqslant 3\max(S_1,S_2,S_3)$.

Case 1. S_1 is maximal. Let

$$u(\alpha;L)=\sum_{x\in\mathscr{A}(L,R)}e(\alpha g(x;wm_1\cdots m_j)),$$
$$v(\alpha)=\sum_{z,\boldsymbol{h}_j}e(\alpha\Psi(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w)).$$

Then

$$S_s(P,Q,R)\ll\int_0^1\left|v(\alpha)^2u(\alpha;M)u(\alpha;Q)^{2s-1}\right|\,d\alpha$$

and an application of Hölder's inequality gives

$$S_s(P,Q,R)\ll\left(S_s(P,M,R)\right)^{1/(2s)}\left(S_s(P,Q,R)\right)^{1-1/(2s)},$$

whence

$$S_s(P,Q,R) \ll S_s(P,M,R) \leqslant M^2 S_{s-1}(P,Q,R).$$

Case 2. S_2 is maximal. Since the variable w plays no role in the estimation of $S_s(P,Q,R)$ in this case, the analysis of [8] may be used without modification, giving

$$S_s(P,Q,R) \ll QMP^{o(1)} S_{s-1}(P,Q,R).$$

Case 3. S_3 is maximal. As in [8], for each solution of (2.1) counted by S_3, there are numbers $c_i|x_i$ and $d_i|y_i$ $(1 \leqslant i \leqslant s)$ with $M < c_i, d_i \leqslant MR$ and $(c_i, \Psi'(z)) = (d_i, \Psi'(z')) = 1$ for each i. Define the generating functions

$$u_d(\alpha) = \sum_{x \in \mathscr{B}(Q/d,R)} e(\alpha g(dx; w m_1 \cdots m_j)),$$

$$v_d(\alpha) = \sum_{\substack{z, \boldsymbol{h}_j \\ (d, \Psi'(z))=1}} e(\alpha \Psi(z; \boldsymbol{h}_j, \boldsymbol{m}_j, w)).$$

Writing $C = c_1 \cdots c_s$ and $D = d_1 \cdots d_s$ and applying Hölder's inequality twice gives

$$\begin{aligned} S_s(P,Q,R) &\ll \sum_{\boldsymbol{c},\boldsymbol{d}} \int_0^1 v_C(\alpha) v_D(\alpha) \prod_{i=1}^s u_{c_i}(\alpha) u_{d_i}(\alpha)\, d\alpha \\ &\leqslant \sum_{\boldsymbol{c},\boldsymbol{d}} \prod_{i=1}^s \left(\int_0^1 |v_C^2 u_{c_i}^{2s}| \right)^{1/2s} \left(\int_0^1 |v_D^2 u_{d_i}^{2s}| \right)^{1/2s} \\ &\leqslant \sum_{\boldsymbol{c},\boldsymbol{d}} \prod_{i=1}^s \left(\int_0^1 |v_{c_i}^2 u_{c_i}^{2s}| \right)^{1/2s} \left(\int_0^1 |v_{d_i}^2 u_{d_i}^{2s}| \right)^{1/2s} \\ &\leqslant \left(\sum_{M<d\leqslant MR} \left(\int_0^1 |v_d^2 u_d^{2s}| \right)^{1/2s} \right)^{2s} \\ &\leqslant (MR)^{2s-1} V, \end{aligned}$$

where V is the number of solutions of

$$\Psi(z; \boldsymbol{h}_j, \boldsymbol{m}_j, w) - \Psi(z'; \boldsymbol{h}_j', \boldsymbol{m}_j, w) + \sum_{i=1}^s (g(dx_i; m_1 \cdots m_j w) - g(dy_i; m_1 \cdots m_j w)) = 0$$

with

$$P/2 \leqslant z, z' \leqslant P; \quad M < d \leqslant MR; \quad (d, \Psi'(z)) = (d, \Psi'(z')) = 1$$
$$\forall i: \quad 1 \leqslant h_i, h_i' \leqslant P/m_i^{k_t}, \quad x_i, y_i \in \mathscr{A}(Q/M, R).$$

We now make the critical observation that

$$g(dx; m_1 \cdots m_j w) = d^{k_t} g(x; m_1 \cdots m_j w d),$$

which implies

$$\Psi(z) \equiv \Psi(z') \pmod{d^{k_t}}.$$

For some constant K, depending only on k, the number of solutions of $\Psi(z) \equiv u \pmod{d^{k_t}}$ with $(d, \Psi'(z)) = 1$ is $\ll d^{K/\log\log d} \ll P^{o(1)}$. As in [8], an application of the Cauchy-Schwarz inequality reduces the problem to the case where $z \equiv z' \pmod{d^{k_t}}$ and $\boldsymbol{h}_j = \boldsymbol{h}'_j$, introducing a factor $HMRP^{o(1)}$, where $H = \prod_{i=1}^{j}(P/m_i^{k_t})$. Writing $m_{j+1} = d$, and recalling (2.2), we obtain

$$S_s(P,Q,R) \ll H(MR)^{2s}P^{o(1)} \max_{M<m_{j+1}\leqslant MR} T_s(P,Q,R,M;\Psi;\boldsymbol{m}_{j+1};w).$$

Combining the three cases yields

$$S_s(P,Q,R) \ll QMP^{o(1)}S_{s-1}(P,Q,R) + H(MR)^{2s}P^{o(1)} \max_{m_{j+1}} T_s(P,Q,R,M).$$

If the second term on the right is the larger, we're done. If the first term dominates, by Hölder's inequality (as in case 1)

$$\begin{aligned} S_{s-1}(P,Q,R) &\ll \left(\int_0^1 |v(\alpha)^2 u(\alpha;Q)^{2s}|\,d\alpha\right)^{1-1/s} \left(\int_0^1 |v(\alpha)|^2\,d\alpha\right)^{1/s} \\ &\ll (S_s(P,Q,R))^{1-1/s}(PH^2)^{1/s}, \end{aligned}$$

thus

$$S_s(P,Q,R) \ll PH^2(QMP^{o(1)})^s. \tag{2.3}$$

On the other hand, counting only the trivial solutions of (2.2), i.e. solutions with $x_i = y_i$ for each i, $z = z'$ and $\boldsymbol{h}_j = \boldsymbol{h}'_j$ gives the lower bound

$$T_s(P,Q,R,M) \geqslant PH|\mathscr{A}(Q/M,R)|^s \gg PH(Q/M)^s.$$

Combined with (2.3) this yields

$$S_s(P,Q,R) \ll HM^{2s}P^{o(1)} \max_{m_{j+1}} T_s(P,Q,R,M;\Psi;\boldsymbol{m}_{j+1};w)$$

and the lemma follows in this case. □

Suppose now that

$$0 < \phi_j \leqslant \frac{1}{k_t} \qquad (1 \leqslant j \leqslant k_t - 1)$$

and for each j set

$$M_j = P^{\phi_j}, \qquad H_j = PM_j^{-k_t}, \qquad \theta_j = \phi_1 + \cdots + \phi_j, \qquad Q_j = P^{1-\theta_j}.$$

As in [8], define the modified forward difference operator Δ_j by

$$\Delta_0(p(z)) = p(z), \qquad \Delta_1(p(z);h_1,m_1) = \frac{1}{m_1^{k_t}}\left(p(z+h_1m_1^{k_t}) - p(z)\right)$$

and

$$\Delta_j(p(z);\boldsymbol{m}_j;\boldsymbol{h}_j) = \frac{1}{(m_1\cdots m_j)^{k_t}} \times \sum_{\substack{\varepsilon_1,\dots,\varepsilon_j\\ \varepsilon_i\in\{0,1\}}} (-1)^{j+\varepsilon_1+\dots+\varepsilon_j} p(z+\varepsilon_1 h_1 m_1^{k_t}+\dots+\varepsilon_j h_j m_j^{k_t})$$

for $j \geqslant 1$. Now define polynomials Ψ_j, in $2j+2$ variables, by

$$\Psi_0(z,w) = g(z;w),$$
$$\Psi_j(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w) = \Delta_j(g(z;w);\boldsymbol{m}_j;\boldsymbol{h}_j) \quad (j \geqslant 1).$$

It is straightforward to show that

$$\Psi_j(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w) = (h_1\cdots h_j) \times \sum_{i=1}^{t} w^{k_i-k_t} a_i \sum_{\boldsymbol{l}} \binom{k_i}{l_0,\cdots,l_j} z^{l_0}(h_1 m_1^{k_t})^{l_1-1}\cdots(h_j m_j^{k_t})^{l_j-1},$$

where the inner sum is over $\boldsymbol{l} = (l_0, l_1, \dots, l_j)$ satisfying $l_0+\dots+l_j = k_i$, $l_0 \geqslant 0$ and $l_i \geqslant 1$ $(i = 1, 2, \dots, j)$. As a polynomial in z the leading term is $a_1 w^{k-k_t} h_1\cdots h_j z^{k-j}$.■ Also, if each $0 < h_i \leqslant P/m_i^{k_t}$, $m_i > 0$, $w > 0$, $P/2 \leqslant z \leqslant P$ and P sufficiently large depending only on f, we have $\Psi_j'(z;\boldsymbol{h}_j,\boldsymbol{m}_j,w) > 0$.

Further define

$$S_{s,j}(\boldsymbol{m}_j) = S_s(P,Q_j,R;\Psi_j;\boldsymbol{m}_j;w)$$

and

$$T_{s,j}(\boldsymbol{m}_{j+1}) = T_s(P,Q_j,R,M_{j+1};\Psi_j;\boldsymbol{m}_{j+1};w).$$

In the following we assume w is fixed and that $M_j \leqslant m_j \leqslant M_j R$ for each j.

By Lemma 2.1 we obtain

$$S_{s,j}(\boldsymbol{m}_j) \ll \left(\prod_{i=1}^{j} H_i\right)(M_{j+1}R)^{2s}P^{o(1)} \max_{M_{j+1}\leqslant m_{j+1}\leqslant M_{j+1}R} T_{s,j}(\boldsymbol{m}_{j+1}).$$

Lemma 2.2. *We have*

$$T_{s,j}(\boldsymbol{m}_{j+1}) \ll P\left(\prod_{i=1}^{j} H_i\right) I_s(Q_{j+1},R;wm_1\cdots m_{j+1}) + \left\{I_s(Q_{j+1},R;wm_1\cdots m_{j+1})S_{s,j+1}(\boldsymbol{m}_{j+1})\right\}^{1/2}.$$

Proof. We have $T_{s,j}(\boldsymbol{m}_{j+1}) \leqslant U_0 + 2U_1$, where U_0 is the number of solutions of (2.2) with $z = z'$ and U_1 is the number of solutions of (2.2) with $z < z'$. Clearly

$$U_0 \leqslant P\left(\prod_{i=1}^{j} H_i\right) I_s(Q_{j+1},R;wm_1\cdots m_{j+1}).$$

To bound U_1, we set

$$z' = z + h_{j+1} m_{j+1}^{k_t}$$

so that $1 \leqslant h_{j+1} \leqslant H_{j+1}$ and

$$m_{j+1}^{-k_t}(\Psi_j(z'; \boldsymbol{h}_j, \boldsymbol{m}_j, w) - \Psi_j(z; \boldsymbol{h}_j, \boldsymbol{m}_j, w)) = \Psi_{j+1}(z; \boldsymbol{h}_{j+1}, \boldsymbol{m}_{j+1}, w).$$

Thus (2.2) becomes

$$\Psi_{j+1}(z; \boldsymbol{h}_{j+1}, \boldsymbol{m}_{j+1}, w) + \sum_{i=1}^{s} (g(x_i; m_1 \cdots m_{j+1} w) - g(y_i; m_1 \cdots m_{j+1} w)) = 0.$$

Write

$$\begin{aligned} v(\alpha) &= \sum_{z, \boldsymbol{h}_{j+1}} e(\alpha \Psi_{j+1}(z; \boldsymbol{h}_{j+1}, \boldsymbol{m}_{j+1}, w)), \\ u(\alpha) &= \sum_{x \in \mathscr{B}(Q_{j+1}, R)} e(\alpha g(x; m_1 \cdots m_{j+1} w)). \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} U_1 &= \int_0^1 v(\alpha) |u(\alpha)|^{2s} \, d\alpha \\ &\leqslant \left(\int_0^1 |u(\alpha)|^{2s} \right)^{1/2} \left(\int_0^1 |v(\alpha)|^2 |u(\alpha)|^{2s} \right)^{1/2} \\ &= \left\{ I_s(P, Q_{j+1}; w m_1 \cdots m_{j+1}) S_{s,j+1}(\boldsymbol{m}_{j+1}) \right\}^{1/2}. \end{aligned}$$

□

Lemma 2.3. *Suppose $\lambda_s = 2s - k_t + \Delta$ is admissible and $\Delta > 0$. Set*

$$\begin{aligned} \theta &= \frac{1}{k_t + \Delta} + \left(\frac{1}{k_t} - \frac{1}{k_t + \Delta} \right) \left(\frac{k_t - \Delta}{2k_t} \right)^{k_t - 1}, \\ \Delta' &= \Delta(1 - \theta) + k_t \theta - 1, \\ \lambda_{s+1} &= 2(s+1) - k_t + \Delta'. \end{aligned}$$

Then λ_{s+1} is admissible.

Proof. When $\Delta = 0$ then $\Delta' = 0$ and the conclusion follows by expressing $I_s(P, R; w)$ as an integral of exponential sums and making a trivial estimate. When $\Delta > 0$ the lemma will follow by iterating Lemmas 2.1 and 2.2 as in [8, Lemma 3.1] using the choice

$$\phi_j = \frac{1}{k_t + \Delta} + \left(\frac{1}{k_t} - \frac{1}{k_t + \Delta} \right) \left(\frac{k_t - \Delta}{2k_t} \right)^{k_t - j} \qquad (1 \leqslant j \leqslant k_t). \tag{2.4}$$

We note that (2.4) and the bound $\Delta \leqslant k_t$ imply that

$$\frac{1}{2k_t} \leqslant \phi_j \leqslant \frac{1}{k_t} \qquad \forall j.$$

Also, since $k \geqslant 3$ and $\Delta > 0$,

$$\theta_{k_t} = 1 - \frac{(k_t-2)\Delta + \Delta^2 + 2\Delta(\frac{1}{2} - \Delta/2k_t)^{k_t-1}}{(k_t+\Delta)^2} < 1.$$

Fix $\varepsilon > 0$. Since λ_s is admissible, there is an $\eta > 0$ so that whenever $R \leqslant Q^\eta$ we have $I_s(Q,R;w) \ll Q^{\lambda_s+\varepsilon}$ uniformly in $w \geqslant 1$. Let η' be a positive real number satisfying

$$\eta' < (1-\theta_{k_t})\eta, \qquad \eta' < \frac{\varepsilon}{4sk_t} \tag{2.5}$$

and put $R = P^{\eta'}$. Then for each j, $I_s(Q_j,R;w) \ll Q_j^{\lambda_s+\varepsilon}$ uniformly in $w \geqslant 1$. We next show by induction on j that

$$T_{s,j}(\boldsymbol{m}_{j+1}) \ll P\left(\prod_{i=1}^{j} H_i\right) Q_{j+1}^{\lambda_s+\varepsilon}, \tag{2.6}$$

where the implied constant may depend on j as well as f, s, ε. First, when $j = k_t - 1$ we have $H_{j+1} = 1$. Representing $S_{s,j+1}(\boldsymbol{m})$ as an integral of exponential sums and taking a trivial bound gives

$$\begin{aligned} S_{s,j+1}(\boldsymbol{m}_{j+1}) &\leqslant P^2\left(\prod_{i=1}^{j} H_i\right)^2 I_s(Q_{j+1},R;wm_1\cdots m_{j+1}) \\ &\ll P^2\left(\prod_{i=1}^{j} H_i\right)^2 Q_{j+1}^{\lambda_s+\varepsilon}. \end{aligned}$$

By Lemma 2.2, (2.6) holds for $j = k_t - 1$. Now suppose $j \leqslant k_t - 2$ and (2.6) holds for all larger j. By Lemma 2.1 and the induction hypothesis,

$$S_{s,j+1}(\boldsymbol{m}_{j+1}) \ll P^{1+o(1)}\left(\prod_{i=1}^{j+1} H_i\right)^2 (M_{j+2}R)^{2s} Q_{j+2}^{\lambda_s+\varepsilon}.$$

Hence, by Lemma 2.2

$$T_{s,j}(\boldsymbol{m}_{j+1}) \ll P\left(\prod_{i=1}^{j} H_i\right) Q_{j+1}^{\lambda_s+\varepsilon}\left\{1 + R^s P^{-1/2+o(1)} H_{j+1} M_{j+2}^s \left(\frac{Q_{j+2}}{Q_{j+1}}\right)^{\frac{1}{2}(\lambda_s+\varepsilon)}\right\}.$$

By (2.4) and (2.5) the term in the braces is $1 + P^{\xi+o(1)}$, where

$$\begin{aligned} \xi &= -\frac{1}{2} + 1 - k_t\phi_{j+1} + s\phi_{j+2} - \frac{1}{2}\phi_{j+2}(\lambda_s+\varepsilon) + \eta' s \\ &= -\frac{1}{2}\varepsilon\phi_{j+2} + \eta' s < 0. \end{aligned}$$

Therefore (2.6) holds for $0 \leqslant j \leqslant k_t - 1$. Finally, by (2.5), (2.6) with $j = 0$ and Lemma 2.1

$$I_{s+1}(P,R;w) \leqslant S_{s,0}(-) \ll (M_1R)^{2s} P^{1+o(1)} Q_1^{\lambda_s+\varepsilon} = P^{L+o(1)},$$

where, since $\theta = \phi_1$,

$$L = 2s\theta + 2s\eta' + 1 + \theta(\lambda_s+\varepsilon) = \lambda_{s+1} + 2s\eta' < \lambda_{s+1} + \varepsilon.$$

Since ε is arbitrary, the lemma follows. □

The admissible values of λ_s arising from Lemma 2.3 are precisely the values of λ_s arising in [8] for the monomial $f(x) = x^{k_t}$. Thus, by the proof of Theorem 2.1 of [9], it follows that these values satisfy $\lambda_s < 2s - k_t + k_t e^{1-2s/k_t}$. Hence we have

Corollary 2.4. *For each $s \geqslant 1$, there is an $\eta = \eta(s) > 0$ such that for $R \leqslant P^\eta$, uniformly in $w \geqslant 1$,*

$$I_s(P,R;w) \ll P^{2s-k_t+k_t e^{1-2s/k_t}}.$$

Remark 1. In Corollary 2.4, the exponent of P tends to $2s-k_t = 2s-k+r$ as $s \to \infty$, which is short by r of the "ideal" exponent $2s-k$ needed in the application to the generalized Waring problem. Fortunately, if r is small, we can overcome this deficiency with a diminishing ranges argument.

Remark 2. We in fact can do a bit better than Corollary 2.4, by taking advantage of the form of $g(x;w)$ when w becomes large. For example, if $w \geqslant c(f,s)Q^k$, the only solutions counted in $I_s(Q,R;w)$ are the "diagonal" solutions corresponding to $x_i = y_i$ for each i and thus $I_s(Q,R;w) = |\mathscr{A}(Q,R)|^s \ll Q^s$. Using such ideas, however, does not appear to break the $2s-k_t$ barrier mentioned in Remark 1, nor increase significantly the rate at which the exponent approaches $2s-k_t$ as s increases. Thus in the application to Waring's problem, only the second order terms will be improved.

3. Application to Waring's Problem

Proof of Theorem 1. Suppose k is large and $r \leqslant k/3$ (else Theorem 1 is no stronger that [10, Theorem 9]). Let n be a large integer, $P = [(n/a_1)^{1/k}]$ and s,u fixed positive integers. For $i = 1,\dots,u$ set

$$P_i = 2^{-i}P^{(1-1/k)^{i-1}}$$

and let

$$Q = \frac{1}{s2^{u+1}}P^{(1-1/k)^u}.$$

Define the generating functions

$$D(\alpha) = \sum_{x=1}^{P} e(\alpha f(x)), \quad F(\alpha) = \sum_{x \in \mathscr{A}(P,R)} e(\alpha f(x)), \quad H(\alpha) = \sum_{x \in \mathscr{A}(Q,R)} e(\alpha f(x))$$

and

$$G(\alpha) = E_1(\alpha)\cdots E_u(\alpha), \qquad E_i(\alpha) = \sum_{P_i < x \leqslant 2P_i} e(\alpha f(x)) \quad (i = 1,\dots,u).$$

Let $R(n)$ denote the number of solutions of

$$n = f(w_1) + f(w_2) + \sum_{i=1}^{6k} f(x_i) + \sum_{i=1}^{2s} f(y_i) + \sum_{i=1}^{u}(f(z_i) + f(z_i')) \tag{3.1}$$

with

$$1 \leqslant w_1, w_2 \leqslant P, \quad \forall i: \quad x_i \in \mathscr{A}(P,R), \quad y_i \in \mathscr{A}(Q,R), \quad P_i < z_i, z_i' \leqslant 2P_i.$$

Then

$$R(n) = \int_0^1 D(\alpha)^2 F(\alpha)^{6k} G(\alpha)^2 H(\alpha)^{2s} e(-n\alpha)\, d\alpha.$$

We now define the major and minor arcs. For $2 \leqslant a \leqslant q \leqslant a_1 P^{1/2} R$, $(a,q)=1$ define

$$\mathfrak{M}(q,a) = \left\{ \alpha \in [0,1] : |\alpha - a/q| \leqslant \frac{P^{1/2}}{qP^k} \right\}.$$

Also let

$$\mathfrak{M}(1,1) = [0, P^{1/2-k}] \cup [1-P^{1/2-k}, 1], \quad \mathfrak{M} = \bigcup_{a,q} \mathfrak{M}(q,a), \quad \mathfrak{m} = [0,1] \backslash \mathfrak{M}.$$

The major arcs are handled as in Wooley [10, §9], where the function $\mathcal{F}_{1,s}(\alpha)$ there is replaced by $G(\alpha)^2 H(\alpha)^{2s}$. This gives

$$\int_{\mathfrak{M}} D(\alpha)^2 F(\alpha)^{6k} G(\alpha)^2 H(\alpha)^{2s} e(-n\alpha)\, d\alpha \\ \gg D(0)^2 F(0)^{6k} G(0)^2 H(0)^{2s} P^{-k} (\mathfrak{S}(n) + o(1)),$$

where $\mathfrak{S}(n) = \mathfrak{S}_{2s+2u+6k+2,f}(n)$. For the minor arcs, we have

$$\left| \int_{\mathfrak{m}} D(\alpha)^2 F(\alpha)^{6k} G(\alpha)^2 H(\alpha)^{2s} e(-n\alpha)\, d\alpha \right| \\ \leqslant P^2 \sup_{\alpha \in \mathfrak{m}} |F(\alpha)|^{6k} \int_0^1 |H(\alpha)^s G(\alpha)|^2 \, d\alpha. \tag{3.2}$$

By [10, Theorem 7], for large k we have

$$\sup_{\alpha \in \mathfrak{m}} |F(\alpha)| \ll P^{1-1/(6tk \log k)}.$$

The integral on the right side of (3.2) counts the number of solutions of

$$\sum_{i=1}^{u} (f(z_i) - f(z_i')) + \sum_{i=1}^{s} (f(y_i) - f(y_i')) = 0$$

with

$$P_i < z_i, z_i' \leqslant 2P_i, \qquad y_i, y_i' \in \mathscr{A}(Q,R).$$

It is straightforward to show that $z_i = z_i'$ for $i = 1, \dots, u$, which is a consequence of our choice of the numbers P_i. By Corollary 2.4 we obtain

$$\begin{aligned} \int_0^1 |H(\alpha)^s G(\alpha)|^2 \, d\alpha &\ll P_1 \cdots P_u \int_0^1 |H(\alpha)|^{2s} \, d\alpha \\ &\ll P_1 \cdots P_u Q^{2s-(k-r)+\Delta} \\ &\ll H(0)^{2s} G(0)^2 P^{-K_1}, \quad K_1 = k - (\Delta + r)(1-1/k)^u \end{aligned}$$

where $\Delta = (k-r)e^{1-2s/(k-r)}$. Therefore, the left side of (3.2) is is

$$\ll D(0)^2 F(0)^{6k} G(0)^2 H(0)^{2s} P^{-K_2}, \quad K_2 = \frac{1}{t\log k} + k - (\Delta + r)e^{-u/k}.$$

We now choose

$$s = \left[\frac{1}{2}(k-r)\log(k/r-1)\right] + 1, \qquad u = [k\log(4rt\log k)] + 1,$$

which implies $\Delta \leqslant er$ and $K_2 \geqslant k + \frac{0.07}{t\log k}$. Therefore, for P large,

$$R(n) \gg D(0)^2 F(0)^{6k} G(0)^2 H(0)^{2s} P^{-k} (\mathfrak{S}(n) + o(1))$$

and hence

$$\bar{G}(f) \leqslant 6k + 2s + 2u + 2 \leqslant k(\log(krt^2) + 2\log\log k + 10).$$

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