Localized large sums of random variables

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Abstract

We study large partial sums, localized with respect to the sums of variances, of a sequence of centered random variables. An application is given to the distribution of prime factors of typical integers.

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Dedicated to the memory of Walter Philipp

1 Introduction

Consider random variables $X_1, X_2, \ldots$ with $\mathbb{E}X_j = 0$ and $\mathbb{E}X_j^2 = \sigma_j^2$. Let

$$S_n = X_1 + \cdots + X_n, \quad s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2,$$

and assume that (a) $s_n \to \infty$ as $n \to \infty$.

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Given a positive function $f_N \geqslant 1 + 1/N$, we are interested in the behavior of

$$I = \liminf_{N\to\infty} \max_{N<s_n^2\leqslant Nf_N} |S_n|/s_n.$$

If we replace $\liminf$ by $\limsup$, it immediately follows from the law of the iterated logarithm that $I = \infty$ almost surely when f_N is bounded. Our results answer a question originally raised, in oral form, by A. Sárközy and for which a partial answer had previously been given by the second author, see Chap. 3 of Oon (2005).

2 Independent random variables

Assume that the X_j are independent. Then $\mathbb{E}S_n^2 = s_n^2$. In addition to condition (a), we will work with two other mild assumptions, (b) $s_{j+1}/s_j \ll 1$ when $s_j > 0$ and (c) for every $\lambda > 0$, there is a constant $c_\lambda > 0$ such that if n is large enough and $s_m^2 > 2s_n^2$, then

$$\mathbb{P}\left(|S_m - S_n| \geqslant \lambda s_m\right) \geqslant c_\lambda.$$

Condition (b) says that no term in S_n dominates the others. Condition (c) follows if the Central Limit Theorem (CLT) holds for the sequence of S_n, since CLT for S_n implies CLT for $S_m - S_n$ as $(m-n) \to \infty$. For example, (c) holds for i.i.d. random variables, under the Lindeberg condition

$$\forall\varepsilon > 0, \quad \lim_{n\to\infty} \sum_{1\leqslant j\leqslant n} \mathbb{E}\left(X_j^2/s_n^2 : |X_j| > \varepsilon s_n\right) = 0$$

and the stronger Lyapunov condition

$$\exists\delta > 0 : \sum_{1\leqslant j\leqslant n} \mathbb{E}|X_j|^{2+\delta} = o(s_n^{2+\delta}).$$

Condition (c) is weaker, however, than CLT.

Theorem 1 *(i) Suppose (a), (b), and $f_N = (\log N)^M$ for some constant $M > 0$. Then $I < \infty$ almost surely.*

(ii) Suppose (a), (b), (c) and $f_N = (\log N)^{\xi(N)}$ with $\xi(N)$ tending monotonically to ∞. Then $I = \infty$ almost surely.

Remark. In the first statement of the theorem we show in fact that almost surely $I \leqslant 15\sqrt{M+1}(\max_{s_j>0} s_{j+1}/s_j)^2$.

Lemma 2 (Kolmogorov's inequality, 1929) *We have*

$$\mathbb{P}(\max_{1\leqslant j\leqslant k} |S_j| \geqslant \lambda s_k) \leqslant 1/\lambda^2 \qquad (k \geqslant 1).$$

Proof of Theorem 1. By (a) and (b), there is a constant D so that $s_{j+1}/s_j \leqslant D$ for all large j. Define

$$h(n) := \max\{k : s_k^2 \leqslant n\} \qquad (n \in \mathbb{N}^*),$$

so that the conditions $N < s_n^2 \leqslant N f_N$ and $h(N) < n \leqslant h(N f_N)$ are equivalent.

We first consider the case when $f_N := (\log N)^M$. Let

$$N_j := j^{(M+3)j}, \quad t(j) := \lfloor (M+1)(\log j)/\log 2 \rfloor, \quad H_j := 2^{t(j)},$$

and

$$U_j := h(N_j), \quad U_{j,t} := h(2^t N_j) \ \ (0 \leqslant t \leqslant t(j)), \quad V_j := h(H_j N_j) = U_{j,t(j)}.$$

It is possible that $U_{j,t+1} = U_{j,t}$ for some t. Note that for large j, $H_j N_j \geqslant N_j f_{N_j}$.

Let k be a constant depending only on M and D. For $j \geqslant 1$ define the events

$$A_j := \{|S_{V_j}| \leqslant s_{U_{j+1}}\},$$
$$B_j := \bigcap_{0 \leqslant t \leqslant t(j)-1} B_{j,t} \ \text{ where } \ B_{j,t} := \left\{ \max_{U_{j+1,t} \leqslant n \leqslant U_{j+1,t+1}} |S_{U_{j+1,t+1}} - S_n| \leqslant k s_{U_{j+1,t}} \right\},$$
$$C_j := \{|S_{U_{j+1}} - S_{V_j}| \leqslant 2 s_{U_{j+1}}\}.$$

By (b) and the definition of $h(N)$, we have

$$D^{-1}\sqrt{2^t N_j} \leqslant s_{U_{j,t}} \leqslant \sqrt{2^t N_j} \tag{1}$$

for all j, t. It follows from Lemma 2 that

$$\mathbb{P}(\overline{A_j}) \leqslant D^2 \frac{H_j N_j}{N_{j+1}} \leqslant \frac{D^2}{j^2}.$$

Thus, $\sum_{j \geqslant 1} \mathbb{P}(\overline{A_j}) < \infty$ and hence almost surely there is a j_0 so that A_j occurs for $j \geqslant j_0$. Applying Lemma 2 again yields

$$\mathbb{P}(\overline{B_{j,t}}) \leqslant \frac{s_{U_{j+1,t+1}}^2 - s_{U_{j+1,t}}^2}{k^2 s_{U_{j+1,t}}^2} \leqslant \frac{D^2 2^{t+1} N_{j+1}}{k^2 2^t N_{j+1}} = \frac{2D^2}{k^2}.$$

If $k = 3D\sqrt{M+1}$, then

$$\mathbb{P}(B_j) \geqslant \left(1 - \frac{2D^2}{k^2}\right)^{t(j)} \geqslant \frac{1}{j^{1/2}}$$

for large j. Also by Lemma 2, $\mathbb{P}(C_j) \geqslant \frac{3}{4}$, and since B_j and C_j are independent,

$$\sum_{j \geqslant 1} \mathbb{P}(B_j C_j) = \infty.$$

Since the events B_jC_j are independent, the Borel–Cantelli lemma implies that almost surely the events B_jC_j occur infinitely often. Thus, the event $A_jB_jC_j$ occurs for an infinite sequence of integers j. Take such a index j, let $n \in [U_{j+1}, V_{j+1}]$ and $U_{j+1,g-1} < n \leqslant U_{j+1,g}$, where $1 \leqslant g \leqslant t(j+1)$. We have by several applications of (1)

$$\begin{aligned}
|S_n| &\leqslant |S_{V_j}| + |S_{U_{j+1}} - S_{V_j}| + \sum_{0\leqslant t\leqslant g-2} |S_{U_{j+1,t}} - S_{U_{j+1,t+1}}| + |S_n - S_{U_{j+1,g-1}}| \\
&\leqslant 3s_{U_{j+1}} + k \sum_{0\leqslant t\leqslant g-1} s_{U_{j+1,t}} \\
&\leqslant \left\{3 + k(1 + 2^{1/2} + \cdots + 2^{(g-1)/2})\right\}\sqrt{N_{j+1}} \\
&\leqslant 5k\sqrt{2^{g-1}N_{j+1}} \\
&\leqslant 5kDs_n = 15D^2(M+1)^{1/2}s_n.
\end{aligned}$$

This completes the proof of part (i) of the theorem, since

$$V_{j+1} \geqslant h(\tfrac{1}{2}j^{M+1}N_j) \geqslant h(N_j \log^M N_j)$$

for large j.

Now suppose $f_N = (\log N)^{\xi(N)}$ with $\xi(N)$ tending monotonically to ∞.

Let $\lambda > 0$ be arbitrary and define $K := 2D^2$. Let N_1^* be so large that $f_{N_1^*} \geqslant K$. For $j \geqslant 1$ let $N_{j+1}^* = N_j^* K^{u(j)}$, where $u(j) := \lfloor \log f_{N_j^*} / \log K \rfloor$. Put

$$U_j^* := h(N_j^*), \quad U_{j,t}^* := h(K^t N_j^*) \ (0 \leqslant t \leqslant u(j)).$$

Let $J_j := [U_j^*, U_{j+1}^*]$ and

$$Y_j := \max_{n\in J_j} |S_n|/s_n.$$

We have

$$u(j) \geqslant 1 \Rightarrow N_{j+1}^* \geqslant KN_j^* \Rightarrow u(j)/\log j \to \infty.$$

Therefore, by (c), if j is sufficiently large then

$$\begin{aligned}
\mathbb{P}(Y_j \leqslant \lambda/2) &\leqslant \prod_{1\leqslant t\leqslant u(j)} \mathbb{P}\left(|S_{U_{j,t}^*} - S_{U_{j,t-1}^*}| \leqslant \tfrac{1}{2}\lambda(s_{U_{j,t}^*} + s_{U_{j,t-1}^*})\right) \\
&\leqslant \prod_{1\leqslant t\leqslant u(j)} \mathbb{P}\left(|S_{U_{j,t}^*} - S_{U_{j,t-1}^*}| \leqslant \lambda\sqrt{K^t N_j^*}\right) \\
&\leqslant (1-c_\lambda)^{u(j)} \leqslant \frac{1}{j^2}.
\end{aligned}$$

Thus

$$\sum_{j\geqslant 1} \mathbb{P}(Y_j \leqslant \lambda/2) < \infty.$$

Almost surely, $Y_k \leqslant \lambda/2$ for only finitely many k.

Theorem 1 has an analog for Brownian motion, which follows from Theorem 1 and the invariance principle.

Theorem 3 *Let $W(t)$ be Brownian motion on $[0,\infty)$. If $f_N = (\log N)^M$ with fixed $M > 0$, then almost surely*

$$I = \liminf_{N\to\infty} \max_{N<t\leqslant Nf_N} \frac{|W(t)|}{\sqrt{t}} < \infty.$$

If $f_N = (\log N)^{\xi(N)}$ with $\xi(N) \to \infty$, then $I = \infty$ almost surely.

Theorem 3 can be proved directly and more swiftly using the methods used to establish Theorem 1. By invariance principles (e.g. Philipp , 1986), one may deduce from Theorem 3 a version of Theorem 1 where stronger hypotheses on the X_j are assumed. As it stands, now, however, Theorem 1 does not follow from Theorem 3.

3 Dependent random variables

The conclusions of Theorem 1 can also be shown to hold for certain sequences of weakly dependent random variables by making use of almost sure invariance principles. We assume that (d) there exists a sequence of i.i.d. normal random variables Y_j with $\mathbb{E}Y_j^2 = \sigma_j^2$, defined on the same probability space as the sequence of X_j, and such that if $Z_n = Y_1 + \cdots + Y_n$, then

$$|S_n - Z_n| = O(s_n) \qquad \text{a.s.}$$

Of course the variables Y_j are dependent on the X_j, but not on each other. Property (d) has been proved for martingale difference sequences, sequences satisfying certain mixing conditions, and lacunary sequences $X_j = \{n_j\omega\}$ with $\inf n_{j+1}/n_j > 1$, ω uniformly distributed in $[0,1]$ and $\{x\}$ is the fractional part of x. See e.g. Philipp (1986) for a survey of such results.

Theorem 4 *(i) Suppose (a), (b), and (d). If $f_N := (\log N)^M$ for some constant $M > 0$, then $I < \infty$ almost surely.*

(ii) Let $\xi(N)$ tend monotonically to ∞ and set $f_N := (\log N)^{\xi(N)}$. Then $I = \infty$ almost surely.

By (d),

$$I = O(1) + \liminf_{N\to\infty} \max_{N<s_n^2\leqslant Nf_N} |Z_n|/s_n,$$

and we apply Theorem 1 to the sequence of Y_j. The variable Z_n is normal with variance s_n^2, hence (c) holds.

4 Prime factors of typical integers

Consider a sequence of independent random variables Y_p, indexed by prime numbers p, such that $\mathbb{P}(Y_p = 1) = 1/p$ and $\mathbb{P}(Y_p = 0) = 1 - 1/p$. We can think of Y_p as modelling whether or not a "random" integer is divisible by p. As $\mathbb{E}Y_p = 1/p$, we form the centered r.v.'s $X_p = Y_p - 1/p$ (we may also define X_j for non-prime j to be zero with probability 1). Let

$$T_n = \sum_{p \leqslant n} Y_p, \qquad S_n = \sum_{p \leqslant n} X_p.$$

We have $\mathbb{E}X_p^2 = (1 - 1/p)/p$, hence by Mertens' estimate

$$s_n^2 = \sum_{p \leqslant n} \frac{1}{p} - \frac{1}{p^2} = \log_2 n + O(1).$$

Here and in the sequel, $\log_k$ denotes, for integer $k \geqslant 2$, the k-fold iterated logarithm. Since $\mathbb{E}|X_p|^3 \leqslant 1/p$, the Lyapunov condition holds with $\delta = 1$. Then (a), (b) and (c) hold, and therefore the conclusion of Theorem 1 holds. Here take $D = \max_{n \geqslant 2} s_{n+1}/s_n$ since $s_1 = 0$.

Let $\omega(m, t)$ denote the number of distinct prime factors of m which are $\leqslant t$. The sequence $\{T_n : n \geqslant 1\}$ mimics well the behavior of the function $\omega(m, n)$ for a "random" m, at least when n is not too close to m. This is known as the Kubilius model. It can be made very precise, see (Elliott , 1979, Ch. 3, especially pp. 119–122) and Tenenbaum (1999) for the sharpest estimate known to date. Suppose r is an integer with $2 \leqslant r \leqslant x$ and $r = x^{1/u}$, $\boldsymbol{\omega}_r(m) = (\omega(m, 1), \ldots, \omega(m, r))$ and suppose Q is any subset of $\mathbb{Z}^r$. Then, given arbitrary $c < 1$, and uniformly in x, r and Q, we have

$$\frac{1}{x}|\{m \leqslant x : \boldsymbol{\omega}_r(m) \in Q\}| = \mathbb{P}\Big((T_1, \ldots, T_r) \in Q\Big) + O\left(x^{-c} + \mathrm{e}^{-u \log u}\right). \tag{2}$$

An analog of Theorem 1, established by parallel estimates, provides via (2) information about localized large values of

$$\varrho(m, t) := |\omega(m, t) - \log_2 t| / \sqrt{\log_2 t}.$$

Theorem 5 *(i) Let $M > 0$ be fixed, $f_N := (\log N)^M$ and put $K := 30D^2\sqrt{M + 1}$. If $g = g(m) \to \infty$ monotonically as $m \to \infty$ in such a way that $g^2 f_{g^2} \leqslant \log_2 m$ for large m, then for a set of integers m of natural density 1,*[2] *we have*

$$\min_{g(m) \leqslant N \leqslant g(m)^2} \ \max_{N < \log_2 t \leqslant N f_N} \varrho(m, t) \leqslant K.$$

[2] A subset $\mathscr{A}$ of $\mathbb{N}^*$ is said to have natural density 1 if $|\mathscr{A} \cap [1, x]| = x + o(x)$ as $x \to \infty$.

(ii) Let $\xi(N) \to \infty$ in such a way that $f_N := (\log N)^{\xi(N)} \leqslant N$. Suppose that $g(m) \to \infty$ monotonically as $m \to \infty$, that $g(m) \leqslant (\log_2 m)^{1/10}$, and let

$$I_m := \min_{\substack{g(m) \leqslant N \\ N f_N \leqslant \log_2 m}} \max_{N \leqslant \log_2 t \leqslant N f_N} \varrho(m,t).$$

Then, $I_m \to \infty$ on a set of integers m of natural density 1.

We follow the proof of Theorem 1. Keeping the notation introduced there, we see that for large J,

$$\mathbb{P}\left(\bigcap_{J \leqslant j \leqslant 3J/2} \overline{A_j B_j C_j}\right) \leqslant \sum_{J \leqslant j \leqslant 3J/2} \frac{D^2}{j^2} + \prod_{J \leqslant j \leqslant 3J/2} \left(1 - \frac{3}{4\sqrt{j}}\right) \ll \frac{1}{J}.$$

For large G, define J by $N_{J+1} < G \leqslant N_{J+2}$. Then $G^{5/3} > N_{\lfloor 3J/2 \rfloor + 2}$ and $J \gg_M (\log G)/\log_2 G$. Thus, for large G,

$$\mathbb{P}\left(\min_{G \leqslant N \leqslant G^{5/3}} \max_{h(N) < n \leqslant h(N f_N)} \frac{|S_n|}{s_n} \leqslant K\right) \geqslant 1 - O\left(\frac{1}{J}\right) \geqslant 1 - O\left(\frac{\log_2 G}{\log G}\right).$$

The direct number theoretic analog of $|S_n|/s_n$ is

$$\widetilde{\varrho}(m,t) := \frac{\left|\omega(m,t) - \sum_{p \leqslant t} 1/p\right|}{\sqrt{\sum_{p \leqslant t} (1 - 1/p)/p}}.$$

By (2), if G is large and $G \leqslant \sqrt{\log_2 x}$ (so that $G^{5/3} f_{G^{5/3}} \leqslant (\log_2 x)^{7/8}$), then

$$\frac{1}{x}\left|\left\{m \leqslant x : \min_{G \leqslant N \leqslant G^{5/3}} \max_{h(N) < n \leqslant h(N f_N)} \widetilde{\varrho}(m,t) \leqslant K\right\}\right| \geqslant 1 - O\left(\frac{\log_2 G}{\log G}\right).$$

Since $\widetilde{\varrho}(m,t) = \varrho(m,t) + O\left(1/\sqrt{\log_2 t}\right)$, the first part of the theorem follows.

The second part is similar. Note that $\omega(n,x) - \omega\Big(n, x^{1/\sqrt{\log_2 x}}\Big) \leqslant \sqrt{\log_2 x}$ for $n \leqslant x$, and, for brevity, write $g = g(\sqrt{x})$. By (2) with $u := \sqrt{\log_2 x}$, we have, for any fixed K and large x,

$$\begin{aligned}
&\frac{1}{x}\left|\left\{m \leqslant x : \min_{\substack{N \geqslant g \\ N f_N \leqslant \log_2 m}} \max_{N < \log_2 t \leqslant N f_N} \widetilde{\varrho}(m,t) \leqslant K\right\}\right| \\
&\qquad \leqslant \frac{1}{x}\left|\left\{\sqrt{x} \leqslant m \leqslant x : \min_{\substack{N \geqslant g \\ N f_N \leqslant \mathscr{L}(x)}} \max_{N \leqslant \log_2 t \leqslant N f_N} \widetilde{\varrho}(t) \leqslant K + 2\right\}\right| + \frac{1}{\sqrt{x}} \\
&\qquad \leqslant \mathbb{P}\left(\inf_{\substack{N \geqslant g \\ N f_N \leqslant \mathscr{L}(x)}} \max_{h(N) < n \leqslant h(N f_N)} \frac{|S_n|}{s_n} \leqslant K + 2\right) + O\Big(\frac{1}{\log_2 x}\Big),
\end{aligned}$$

where $\mathscr{L}(x) := \log_2 x - \frac{1}{2}\log_3 x$. Since $f_N \leqslant N$, we have $N^*_{j+1} \leqslant (N^*_j)^2$ in the notation of the proof of Theorem 1. The interval

$$\Big[(\log_2 x)^{1/10}, \mathscr{L}(x)^{1/2}\Big]$$

therefore contains at least one interval J_j. By the proof of Theorem 1, for large x, the probability above does not exceed $\sum_{j \geqslant j_0} 1/j^2 \leqslant 1/(j_0 - 1)$, where $j_0 \to \infty$ as $x \to \infty$.

Remarks. The upper bound g^2 of N in the first part can be sharpened. By the same methods, similar results can be proved for a wide class of additive arithmetic functions $r(m,t) = \sum_{p^a \| m} r(p^a)$ in place of $\omega(m,t)$.

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