ON GROUPS WITH PERFECT ORDER SUBSETS

KEVIN FORD, SERGEI KONYAGIN, AND FLORIAN LUCA

ABSTRACT. A finite group G is said to have *Perfect Order Subsets* if for every d, the number of elements of G of order d (if there are any) divides $|G|$. Answering a question of Finch and Jones from 2002, we prove that if G is Abelian, then such a group has order divisible by 3 except in the case $G = \mathbb{Z}/2\mathbb{Z}$. We also place additional restrictions on the order of such groups.

1 Introduction

Consider the multiplicative function

$$f(n) = \prod_{p^a \| n} (p^a - 1).$$

A finite group G is said to have *Perfect Order Subsets* if for every d, the number of elements of G of order d (if there are any) divides $|G|$. This notion was introduced in the paper [1] by C. Finch and L. Jones. In the case of finite Abelian groups, the authors reduced the problem of which groups have this property to the case of groups of the form $G = \prod_{i=1}^{k} (\mathbb{Z}/p_i\mathbb{Z})^{a_i}$, where p_i are primes and $a_i \geqslant 1$. For these groups, it follows from results in [1] that G has *Perfect Order Subsets* if and only if $f(n)|n$. Only 11 examples of such n are known, given below, and only one of these is divisible by the square of an odd prime.

2
$2 \cdot 3$
$2^2 \cdot 3$
$2^3 \cdot 3 \cdot 7$
$2^4 \cdot 3 \cdot 5$
$2^5 \cdot 3 \cdot 5 \cdot 31$
$2^8 \cdot 3 \cdot 5 \cdot 17$
$2^{16} \cdot 3 \cdot 5 \cdot 17 \cdot 257$
$2^{17} \cdot 3 \cdot 5 \cdot 17 \cdot 257 \cdot 131071$
$2^{32} \cdot 3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537$
$2^{11} \cdot 3 \cdot 5 \cdot 11^2 \cdot 23 \cdot 89$

The authors of [1] asked several basic questions about such groups. One of which asks if $|G|$ is not a power of 2, must 3 divide $|G|$? We prove that this is the case for Abelian groups.

Theorem 1. *If $f(n)|n$ and $n > 2$, then $3|n$.*

We also show that $f(n)|n$ implies that $n/f(n)$ is bounded. Note that the divergence of $\prod_p (1 - 1/p)^{-1}$ implies that $n/f(n)$ is unbounded for general n. On the other hand, all of the known examples of n such that $n > 6$ and $f(n)|n$ (given in [1]) satisfy $n = 2f(n)$.

Theorem 2. *For any $n \in \mathbb{N}$, if $f(n)|n$, then $n/f(n) \leqslant 85$.*

The most important property of numbers n with $f(n)|n$ is given by the following easy proposition.

Proposition 1. *If $f(n)|n$, then for every prime $p|n$, every prime divisor of $p - 1$ also divides n.*

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By Proposition 1, knowing that $3 \nmid n$ allows us to exclude many possible prime factors of n. Inductively, define a set $\mathscr{P}$ or primes as follows: (i) $2 \in \mathscr{P}$, (ii) $3 \notin \mathscr{P}$, (iii) for every prime $p \geqslant 5$, $p \in \mathscr{P}$ if and only if all prime factors of $p-1$ are in $\mathscr{P}$. Thus,

$$\mathscr{P} = \{2, 5, 11, 17, 23, 41, 47, 83, 89, 101, 137, 167, 179, 251, 257, 353, 359, 401, 461, 503, \ldots\}. \tag{1.1}$$

By Proposition 1, every prime dividing n must come from $\mathscr{P}$. The set $\mathscr{P}$ has a alternative interpretation as the set of all primes whose *Pratt tree* (see [3]) does not contain a node labeled 3.

Our proof of Theorem 1 is primarily based on the lower bound in the following estimate:

Theorem 3. *We have*

$$0.2512 \leqslant \prod_{p \in \mathscr{P}} \left(1 - \frac{1}{p}\right) \leqslant 0.2793.$$

Since $\mathscr{P}$ omits all primes $p \equiv 1 \pmod 3$, and hence omits all primes q such that $q-1$ has a prime factor which is $1 \pmod 3$, standard application of sieve methods yields the upper bound

$$\mathscr{P}(x) := \#\{p \leqslant x : p \in \mathscr{P}\} \ll \frac{x}{(\log x)^{3/2}}.$$

From this one obtains immediately from partial summation that the product in Theorem 3 converges. Obtaining good numerical bounds requires more work.

2 Number theory tools

Our first result is an estimate of Rosser and Schoenfeld [4].

Lemma 2.1. *For any $x > 1$,*

$$\left(1 + \frac{1}{\log^2 x}\right)^{-1} \leqslant e^{\gamma} (\log x) \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \leqslant \left(1 + \frac{1}{\log^2 x}\right).$$

The following general sieve estimate is Theorem 1 of [2].

Lemma 2.2. *Let S be a set of primes containing 2, and put*

$$H(t) = \prod_{\substack{p \leqslant t \\ p \in S}} \left(1 - \frac{1}{p}\right).$$

Then $N(x)$, the number of primes $p \leqslant x$ such that all the prime factors of $p-1$ are in S, satisfies

$$N(x) \leqslant \frac{x}{(1 + 1/\log x) I(x)}, \qquad I(x) = \int_1^{\sqrt{x}} \frac{\log t}{t} H(t)\, dt.$$

Lemma 2.3. *Let S be any set of primes with the property that for all $p \in S$ and prime $q | (p-1)$, $q \in S$. For any $x \geqslant 2$,*

$$\prod_{p \in S} \left(1 - \frac{1}{p}\right) \geqslant \prod_{\substack{p \leqslant x \\ p \in S}} \left(1 - \frac{1}{p}\right) - \frac{8}{\log x}.$$

Proof. We may assume S is nonempty, so $2 \in S$. Write $S(x) = \#\{p \in S : p \leqslant x\}$. For $j \geqslant 0$, let $y_j = x^{2^j}$ and

$$H_j = \prod_{\substack{p \leqslant y_j \\ p \in S}} \left(1 - \frac{1}{p}\right).$$

Without loss of generality, suppose $H_0 > 8/\log x$, so in particular $x \geqslant e^{16}$. We derive by induction lower estimates for H_j. By Lemma 2.2, when $y_{j-1} \leqslant t \leqslant y_j$, we have

$$S(t) \leqslant \frac{8t}{(1+1/\log t)H_{j-1}\log^2 t}.$$

By partial summation and the inequality $\log(1-\frac{1}{t}) \geqslant -\frac{1}{t-1}$,

$$\begin{aligned}\log\left(\frac{H_j}{H_{j-1}}\right) &= \sum_{\substack{y_{j-1}<p\leqslant y_j\\ p\in S}} \log\left(1-\frac{1}{p}\right)\\ &= S(y_j)\log\left(1-\frac{1}{y_j}\right) - S(y_{j-1})\log\left(1-\frac{1}{y_{j-1}}\right) - \int_{y_{j-1}}^{y_j} \frac{S(u)}{u^2-u}\,du\\ &\geqslant -\frac{S(y_j)}{y_j-1} - \frac{y_j}{y_j-1}\int_{y_{j-1}}^{y_j}\frac{S(u)}{u^2}\,du\\ &\geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\left(\frac{1}{\log^2 y_j+\log y_j} + \int_{y_{j-1}}^{y_j}\frac{du}{u(\log^2 u+\log u)}\right).\end{aligned}$$

Using the relation $y_j = y_{j-1}^2$, we find that the integral above equals $\log(1+\frac{1}{\log y_j+1})$. Now $\log(1+\varepsilon) \leqslant \varepsilon - \frac{1}{3}\varepsilon^2$ for $\varepsilon = \frac{1}{\log y_j+1} \leqslant \frac{1}{3}$. Thus,

$$\begin{aligned}\log\left(\frac{H_j}{H_{j-1}}\right) &\geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\left(\frac{1}{\log^2 y_j+\log y_j} + \frac{1}{\log y_j+1} - \frac{1}{3(\log y_j+1)^2}\right)\\ &= -\frac{8}{H_{j-1}\log y_j}\left(\frac{y_j}{y_j-1}\right)\left(1-\frac{\log y_j}{3(\log y_j+1)^2}\right).\end{aligned}$$

Since $y_j \geqslant y_0 \geqslant e^{16}$, the right side above is $\geqslant -8/(H_{j-1}\log y_j)$. Therefore,

$$H_j \geqslant H_{j-1}\exp\left\{-\frac{8}{H_{j-1}\log y_j}\right\} \geqslant H_{j-1} - \frac{8}{\log y_j} = H_{j-1} - \frac{8\cdot 2^{-j}}{\log x}.$$

Iterating this inequality concludes the proof. □

3 Proof of Theorem 1

Before describing these, we show how to deduce Theorem 1 from Theorem 3. Observe that

$$\frac{f(n)}{n} = \prod_{p^a \| n}\left(1-\frac{1}{p^a}\right). \tag{3.1}$$

Suppose that $f(n)|n$ and $3\nmid n$. If $2^9|n$, then (3.1) and Theorem 3 imply

$$\frac{f(n)}{n} \geqslant \frac{511}{512}\prod_{\substack{p\geqslant 5\\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right) \geqslant \frac{511}{512}(0.5024) > \frac{1}{2}.$$

Hence, $f(n)\nmid n$. Thus, $2^k\|n$, where $1\leqslant k\leqslant 8$. If $k=1$, then $4\nmid f(n)$, which means that $n=2p^a$ for some odd prime p. But then (3.1) and $p\geqslant 5$ imply

$$2 < \frac{n}{f(n)} = \frac{2}{1-1/p^a} < 3,$$

so that $f(n) \nmid n$. If k is even, then $3|(2^k-1)|f(n)|n$, a contradiction. Finally, if $k \in \{3,5,7\}$, then n has at most 7 odd prime factors, hence

$$\frac{f(n)}{n} \geqslant \frac{7}{8} \prod_{\substack{5 \leqslant p \leqslant 83 \\ p \in \mathscr{P}}} \left(1 - \frac{1}{p}\right) > \frac{1}{2},$$

so $f(n) \nmid n$. Therefore, $f(n)|n$ implies $3|n$.

Proof of Theorem 3. The proof has two parts. The first is a computer calculation of all of the elements of $\mathscr{P}$ which are less than

$$x_0 = 2^{44} \approx 1.76 \times 10^{13},$$

consisting of 39479071 primes. This computation took about 120 hours on the first authors' desktop computer. Rather than compute the elements of $\mathscr{P}$ one by one, the algorithm sieved a large interval of integers $(A, B]$ (size about 10^8), both sieving out the residue classes $0 \pmod p$ for primes $\leqslant \sqrt{B}$, but also sieving the residue classes $1 \pmod p$ for primes $p \in \mathscr{P}$, $p \leqslant B/2$. Stopping the computation at a power of 2 was convenient for the second part of the proof – using the results of the computation to estimate $\mathscr{P}(x)$ for $x > x_0$.

Lemma 3.1. *Let $x_0 = 2^{44}$. Then*

$$\prod_{\substack{p \in \mathscr{P} \\ p \leqslant x_0}} \left(1 - \frac{1}{p}\right) = 0.27923438887\ldots$$

Furthermore, with $s = 0.6$ we have

$$\mathscr{P}(x) \leqslant \begin{cases} \alpha x^s + 2 & (2^9 \leqslant x \leqslant x_0), \alpha = 0.445836183, \\ \alpha' x^s + 2 & (x \leqslant x_0), \alpha' = 0.501761301. \end{cases}$$

Estimating accurately $\mathscr{P}(x)$ is likely a very hard problem. It appears that $\mathscr{P}(x) \approx x^{5/8}$.

Conjecture 1. *For some $c > 0$, $\mathscr{P}(x) \ll x^{1-c}$.*

Note that if $p \in \mathscr{P}$, then $p \equiv 2 \pmod 3$, hence $\Omega(p-1)$ is even. A second computer program was used to generate even numbers which are products of primes in $\mathscr{P}$. Specifically, let

$$\mathscr{N}^- = \{n : 2|n, P^+(n) \leqslant x_0, \Omega(n) \text{ odd}, p|n \implies p \in \mathscr{P}\} = \{2, 8, 20, 32, 44, \ldots\},$$
$$\mathscr{N}^+ = \{n : 2|n, P^+(n) \leqslant x_0, \Omega(n) \text{ even}, p|n \implies p \in \mathscr{P}\} = \{4, 10, 16, 22, 34, \ldots\},$$

and, setting $\delta = \frac{1}{10}$, let

$$h_j^- = \sum_{\substack{n \in \mathscr{N}^- \\ n < 2^{j\delta}}} \frac{1}{n^s}.$$

If $n \in \mathscr{N}^\pm$ and the odd part of n is given, then the parity of the exponent of 2 in the prime factorization of n is fixed. Thus,

$$\sum_{\substack{n \in \mathscr{N}^\pm \\ P^+(n) < 2^{j\delta}}} \frac{1}{n^s} \leqslant g_j := \frac{2^{-s}}{1 - 4^{-s}} \prod_{\substack{p \in \mathscr{P} \\ 2 < p < 2^{j\delta}}} \left(1 - p^{-s}\right)^{-1}. \tag{3.2}$$

The elements of $\mathscr{N}^-$ were computed exactly up to 2^{36}. Our next task is to use this data to obtain crude upper bounds on $\mathscr{P}(x)$ in the range $x_0 < x \leqslant 2^{72}$:

Lemma 3.2. *Let $\delta = \frac{1}{10}$ and $s = 0.6$. For every integer j satisfying $44 < j\delta \leqslant 72$, we have*

$$\mathscr{P}(x) \leqslant C_j x^s \qquad (2^{(j-1)\delta} < x \leqslant 2^{j\delta}),$$

where

$$C_j = \frac{72}{2^{(j-1)\delta s}} + \min_{\max(9, j\delta - 44) \leqslant t\delta \leqslant 44} \Big[\alpha' \left(g_t - h^-_{\min(t, j-1-t)} \right) + \alpha \left(h^-_{j-t} - h^-_{j-1-44/\delta} \right) \Big] + \sum_{i=1+44/\delta}^{j-1} C_i \left(h^-_{j+1-i} - h^-_{j-i} \right).$$

Moreover, the sequence (C_j) is increasing.

Proof. We proceed by induction on j. Suppose $\delta j > 44$ and the given bounds have been proved for $x_0 < x \leqslant 2^{(j-1)\delta}$. Let $\max(9, j\delta - 44) \leqslant t\delta \leqslant 44$ and put $y = 2^{t\delta}$. Suppose that $2^{(j-1)\delta} < x \leqslant 2^{j\delta}$. Suppose that $p \in \mathscr{P}$ with $p \leqslant x$, let $q = P^+(p-1)$ and $p - 1 = qn$. Then $P^+(n) \leqslant \min(q, x/q) \leqslant x_0$, so $n \in \mathscr{N}^-$. We have (i) $q \leqslant 5$, (ii) $q > 5$ and $n \geqslant x/y$, (iii) $q > 5$ and $x/x_0 \leqslant n < x/y$, or (iv) $q > 5$ and $n < x/x_0$. In case (i), $p - 1$ is a power of two (there are exactly 4 such p) or $p - 1 = 2^a 5^b$ with $a \geqslant 1$, $b \geqslant 1$ (there are 68 such primes $p \leqslant 2^{72}$). Now let $\mathscr{P}^*(x) = \mathscr{P}(x) - 2$. Using (3.2), the number of primes counted in case (ii) is at most

$$\begin{aligned} \sum_{\substack{n \in \mathscr{N}^- \\ x/y \leqslant n < x \\ P^+(n) \leqslant y}} \mathscr{P}^* \left(\frac{x}{n} \right) &\leqslant \alpha' x^s \sum_{\substack{n \in \mathscr{N}^- \\ x/y \leqslant n < x \\ P^+(n) \leqslant y}} \frac{1}{n^s} \\ &\leqslant \alpha' x^s \Big(\sum_{\substack{n \in \mathscr{N}^- \\ P^+(n) \leqslant y}} \frac{1}{n^s} - \sum_{\substack{n \in \mathscr{N}^- \\ n < \min(y, x/y)}} \frac{1}{n^s} \Big) \\ &\leqslant \alpha' x^s \left(g_t - h^-_{\min(t, j-1-t)} \right). \end{aligned}$$

In case (iii), $q \leqslant x_0$, hence the number of such p is bounded above by

$$\sum_{\substack{n \in \mathscr{N}^- \\ x/x_0 \leqslant n < x/y}} \mathscr{P}^* \left(\frac{x}{n} \right) \leqslant \alpha x^s \sum_{\substack{n \in \mathscr{N}^- \\ x/x_0 \leqslant n < x/y}} \frac{1}{n^s} \leqslant \alpha x^s \left(h^-_{j-t} - h^-_{j-1-44/\delta} \right).$$

In the final case, we use the induction hypothesis, in particular the supposition that $C_{j-1} > C_{j-2} > \cdots$. Thus, the number of primes counted in case (iv) is at most

$$\begin{aligned} \sum_{\substack{n \in \mathscr{N}^- \\ n < x/x_0}} \mathscr{P}^* \left(\frac{x}{n} \right) &\leqslant \sum_{\substack{n \in \mathscr{N}^- \\ n < x/x_0}} C_i \left(\frac{x}{n} \right)^s, \qquad i = \left\lceil \frac{\log x/n}{\delta \log 2} \right\rceil \\ &\leqslant \sum_{i=44/\delta+1}^{j-1} x^s C_i \sum_{\substack{n \in \mathscr{N}^- \\ 2^{(j-i)\delta} \leqslant n < 2^{(j-i+1)\delta}}} \frac{1}{n^s} \\ &\leqslant x^s \sum_{i=44/\delta+1}^{j-1} C_i \left(h^-_{j+1-i} - h^-_{j-i} \right). \end{aligned}$$

Combining the estimates in cases (i)–(iv) proves the given assertion in the range $2^{(j-1)\delta} < x \leqslant 2^{j\delta}$. The monotonicity of the sequence (C_j) follows by direct calculation. $\square$

We now develop bounds on $\mathscr{P}(x)$ for $x > 2^{72}$. Let

$$N^- = \sum_{n \in \mathscr{N}^-} \frac{1}{n}, \qquad N^+ = \sum_{n \in \mathscr{N}^+} \frac{1}{n}.$$

By direct application of the computed elements of $\mathscr{P}$ which are $\leqslant x_0$, we obtain

$$N^+ + N^- = \frac{1}{2} \prod_{\substack{p \leqslant x_0 \\ p \in \mathscr{P}}} \left(1 - \frac{1}{p}\right)^{-1} = 1.790610\ldots$$

and

$$N^+ - N^- = \sum_{n \in \mathscr{N}^+ \cup \mathscr{N}^-} \frac{(-1)^{\Omega(n)}}{n} = -\frac{1}{2} \prod_{\substack{p \leqslant x_0 \\ p \in \mathscr{P}}} \left(1 + \frac{1}{p}\right)^{-1} = -0.1968977\ldots.$$

Thus,

$$N^- \leqslant 0.993755, \qquad N^+ \leqslant 0.796857. \tag{3.3}$$

Primarily due to the fact that N^- is so close to 1, our bounds from now on take the shape

$$\mathscr{P}(x) \leqslant K_i x \qquad (2^{i-1} < x \leqslant 2^i). \tag{3.4}$$

First, using the values of C_j from Lemma 3.2, we obtain (3.4) for $45 \leqslant i \leqslant 72$, where

$$K_i = \max_{(i-1)/\delta < j \leqslant i/\delta} C_j \left(2^{(j-1)\delta}\right)^{s-1}.$$

For convenience, define

$$K_i^* = \max\left(K_{45}, \ldots, K_i\right).$$

Lemma 3.3. *For $i \geqslant 73$, we have (3.4), where*

$$K_i = (2^{i-1})^{s-1} g_{44/\delta} + \frac{1}{x_0} + \frac{K_{i-1}}{2} + \frac{K_{i-3}}{8} + \left(N^- - 5/8\right) K_{i-4}^*$$
$$+ \sum_{2 \leqslant k \leqslant (i-2)/44} \frac{\left(K_{i-44(k-1)}^*\right)^k}{k!} N_k \left(1 + (i - 44k)\log 2\right)^{k-1},$$

where

$$N_k = \begin{cases} N^+ & k \text{ even} \\ N^- & k \text{ odd}. \end{cases}$$

Proof. Again, we use induction on i. Suppose that $2^{i-1} < x \leqslant 2^i$. If $p \in \mathscr{P}$, then $p \equiv 2 \pmod 3$. Thus, if $P^+(p-1) \leqslant x_0$ then $p - 1 \in \mathscr{N}^+$. Hence, the number of $p \leqslant x$ with $p \in \mathscr{P}$ and $P^+(p-1) \leqslant x_0$ is at most

$$\sum_{\substack{n \leqslant x-1 \\ n \in \mathscr{N}^+}} \left(\frac{x}{n}\right)^s \leqslant x^s g_{44/\delta}.$$

The number of $p - 1$ divisible by the square of a prime $> x_0$ is trivially at most

$$\sum_{q > x_0} \frac{x}{q^2} \leqslant \frac{x}{x_0}.$$

If $P^+(p-1) > x_0$ and $p-1$ is not divisible by the square of any prime $> x_0$, let k be the number of prime factors of $p-1$ which are $> x_0$. Using the fact that the smallest 3 elements of $\mathscr{N}^-$ are $2, 8, 20$, the number of p with $k=1$ is at most

$$\sum_{\substack{n\in\mathscr{N}^- \\ n<x/x_0}} \mathscr{P}\left(\frac{x}{n}\right) \leqslant \mathscr{P}\left(\frac{x}{2}\right) + \mathscr{P}\left(\frac{x}{8}\right) + \sum_{\substack{n\in\mathscr{N}^- \\ 20\leqslant n\leqslant x/x_0}} \mathscr{P}\left(\frac{x}{n}\right)$$
$$\leqslant \frac{x}{2}K_{i-1} + \frac{x}{8}K_{i-3} + (N^- - 5/8)K^*_{i-4}.$$

Now suppose $k \geqslant 2$ and put $\mathscr{N}_k = \mathscr{N}^-$ if k is odd and $\mathscr{N}_k = \mathscr{N}^+$ if k is even. Observe that $i \geqslant 44k$. As there are $k!$ was to order the prime factors of $p-1$ which are $> x_0$, the number of $p \leqslant x$ corresponding to this value of k is at most

$$\frac{1}{k!}\sum_{\substack{n\in\mathscr{N}_k \\ n<x/x_0^k}} \sum_{\substack{x_0<q_1\leqslant x/(nx_0^{k-1}) \\ q_1\in\mathscr{P}}} \cdots \sum_{\substack{x_0<q_{k-1}\leqslant x/(nx_0^{k-1}) \\ q_{k-1}\in\mathscr{P}}} \mathscr{P}\left(\frac{x}{nq_1\cdots q_{k-1}}\right)$$
$$\leqslant \frac{K^*_{i-44(k-1)}}{k!} x \sum_{n\in\mathscr{N}_k} \frac{1}{n}\Big(\sum_{\substack{x_0<q\leqslant x/x_0^{k-1} \\ q\in\mathscr{P}}} \frac{1}{q}\Big)^{k-1}$$
$$\leqslant \frac{K^*_{i-44(k-1)}}{k!} x N_k \Big(\frac{\mathscr{P}(x/x_0^{k-1})}{x/x_0^{k-1}} + \int_{x_0}^{x/x_0^{k-1}} \frac{\mathscr{P}(u)}{u^2}\,du\Big)^{k-1}$$
$$\leqslant \left(K^*_{i-44(k-1)}\right)^k \frac{x}{k!} N_k \left(1+\log(x/x_0^k)\right)^{k-1}. \qquad \square$$

Heuristically, the terms in the sum corresponding to $k=1$ dominate the others. These terms total at most $K^*_{i-1}N^- < K^*_{i-1}$, which means that the sequence (K_i) changes very slowly with i. In fact, $K_i \leqslant 0.0001407$ for $45 \leqslant i \leqslant 640$. Using computed values of K_i for $i \leqslant 640$, we obtain, with $x_1 = 2^{640}$,

$$\prod_{\substack{p\leqslant x_1 \\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right) \geqslant \prod_{\substack{p\leqslant x_0 \\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right)\exp\left\{-\sum_{p>x_0}\frac{1}{p^2} - \sum_{\substack{x_0<p\leqslant x_1 \\ p\in\mathscr{P}}}\frac{1}{p}\right\}$$
$$\geqslant \prod_{\substack{p\leqslant x_0 \\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right)\exp\left\{-\frac{1}{x_0} - \frac{\mathscr{P}(x_1)}{x_1} + \frac{\mathscr{P}(x_0)}{x_0} - \int_{x_0}^{x_1}\frac{\mathscr{P}(u)}{u^2}\,du\right\} \tag{3.5}$$
$$\geqslant \prod_{\substack{p\leqslant x_0 \\ p\in\mathscr{P}}}\left(1-\frac{1}{p}\right)\exp\left\{\frac{39479070}{x_0} - K_{640} - \sum_{i=45}^{640} K_i \log 2\right\}$$
$$\geqslant 0.2693.$$

To finish the proof of Theorem 3, take $S=\mathscr{P}$ and $x=x_1=2^{640}$ in Lemma 2.3, and use (3.5). $\square$

4 Proof of Theorem 2

Proposition 2. *Suppose $f(n)|n$ and $n/f(n) \geqslant 5$. Then $\omega(n) \geqslant 46$ and $2^{45}|n$.*

Proof. If $2\|n$ and $n>2$, then $n=2p^b$ for a prime p, so $(p^b-1)|(2p^b)$ and hence $(p^b-1)|2$. This implies $p=3$ and $n=6$. If $2^2|n$ and $2^6\nmid n$, then n has at most 6 odd prime factors and

$$\frac{n}{f(n)} \leqslant \frac{4}{3}\prod_{3\leqslant p\leqslant 13}\frac{p}{p-1} < 4.$$

Now assume $2^6|n$. If $\omega(n) \leqslant 45$, then

$$\frac{f(n)}{n} \geqslant \frac{63}{64} \prod_{3 \leqslant p \leqslant 200} \left(1 - \frac{1}{p}\right) > \frac{1}{5}.$$

Hence, $\omega(n) \geqslant 46$, and thus n has at least 45 odd prime factors. This implies that $2^{45}|f(n)|n$. □

We first prove the following result about primes dividing n to a small power.

Theorem 4. *If $f(n)|n$ and $Q = \{p|n : p^{40} \nmid n\}$, then*

$$\prod_{q \in Q} \left(1 - \frac{1}{q}\right)^{-1} \leqslant 85.32.$$

Proof. By Proposition 2, we may assume $2^{45}|n$, so that $2 \notin Q$. Let t_0 be the smallest prime that

$$\prod_{\substack{p \leqslant t_0 \\ p \in Q}} \left(1 - \frac{1}{p}\right)^{-1} \geqslant 16.016 e^{\gamma}. \tag{4.1}$$

If no such t_0 exists, then the theorem follows, since $16.016e^{\gamma} < 30$. Next, Lemma 2.1 implies

$$\frac{1}{32.032 e^{\gamma}} \geqslant \prod_{p \leqslant t_0} \left(1 - \frac{1}{p}\right) \geqslant \left(1 + \frac{1}{\log^2 t_0}\right)^{-1} \frac{e^{-\gamma}}{\log t_0},$$

which implies that $t_0 \geqslant e^{32}$.

Let $S = \{p : p|n\}$ and $S(x) = \#\{p \leqslant x : p \in S\}$. For any prime q with $q^b \| n$, there are at most b primes $p|n$ with $p \equiv 1 \pmod q$. Hence, by Lemma 2.2, for $x \geqslant t_0$ we have

$$\begin{aligned} S(x) &\leqslant S(\sqrt{x}) + \sum_{\substack{q \in Q \\ q \leqslant \sqrt{x}}} \sum_{\substack{p|n \\ p \equiv 1 \pmod q}} 1 + \#\{\sqrt{x} < p \leqslant x : \forall q \leqslant \sqrt{x} \text{ with } q \in Q, q \nmid (p-1)\} \\ &\leqslant 40 S(\sqrt{x}) + \#\{\sqrt{x} < p \leqslant x : \forall q \leqslant \sqrt{x} \text{ with } q \in Q, q \nmid (p-1)\} \\ &\leqslant 20\sqrt{x} + \frac{x}{(1 + 1/\log x) I(x)}, \end{aligned}$$

where

$$I(x) = \int_1^{\sqrt{x}} H(t) \frac{\log t}{t}\, dt, \qquad H(t) = \prod_{\substack{p \leqslant t \\ p^{40}|n}} \left(1 - \frac{1}{p}\right).$$

By Lemma 2.1 and (4.1),

$$H(t) \geqslant \prod_{p \leqslant \max(t_0, t)} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \leqslant t_0 \\ p \in Q}} \left(1 - \frac{1}{p}\right)^{-1} \geqslant \frac{16.016}{\log \max(t, t_0)} \left(1 + \frac{1}{\log^2 t_0}\right)^{-1} \geqslant \frac{16}{\log \max(t, t_0)}.$$

Hence,

$$I(x) \geqslant \begin{cases} \frac{2 \log^2 x}{\log t_0} & (x \leqslant t_0^2) \\ 8 \log(x/t_0) & (x > t_0^2). \end{cases}$$

Since $\sqrt{x} \leqslant \frac{x}{8000 \log^2 x}$ for $x \geqslant t_0$, we obtain

$$S(x) \leqslant \begin{cases} \frac{x \log t_0}{2 \log^2 x} & (t_0 \leqslant x \leqslant t_0^2) \\ \frac{x}{8 \log(x/t_0)} & (x > t_0^2). \end{cases} \tag{4.2}$$

Note that by (4.1), $S(t_0) \geqslant 1$. By (4.2) and partial summation, if $t = t_0^{C+1} \geqslant t_0^2$ then

$$\prod_{\substack{p\in S\\ t_0<p\leqslant t}} \left(1-\frac{1}{p}\right) \geqslant \exp\Bigg\{-\sum_{\substack{p\in S\\ t_0<p\leqslant t}} \frac{1}{p} - \sum_{p>t_0} \frac{1}{p^2}\Bigg\}$$
$$\geqslant \exp\left\{-\frac{1}{t_0} + \frac{S(t_0)}{t_0} - \frac{S(t)}{t} - \int_{t_0}^{t} \frac{S(u)}{u^2}\,du\right\}$$
$$\geqslant \exp\left\{-\frac{1}{8C\log t_0} - \frac{1}{4} - \frac{1}{8}\log C\right\}.$$

Applying Lemma 2.3 gives

$$\prod_{p\in S} \left(1-\frac{1}{p}\right) \geqslant \prod_{\substack{p\in S\\ p\leqslant t}} \left(1-\frac{1}{p}\right) - \frac{8}{\log t}$$
$$\geqslant \prod_{\substack{p\in S\\ p\leqslant t_0}} \left(1-\frac{1}{p}\right) \cdot \exp\left\{-\frac{1}{8C\log t_0} - \frac{1}{4} - \frac{1}{8}\log C\right\} - \frac{8}{(C+1)\log t_0}.$$

By Lemma 2.1, we obtain the bound

$$\prod_{\substack{p\in Q\\ p>t_0}} \left(1-\frac{1}{p}\right) \geqslant \prod_{\substack{p\in S\\ p>t_0}} \left(1-\frac{1}{p}\right)$$
$$\geqslant \exp\left\{-\frac{1}{8C\log t_0} - \frac{1}{4} - \frac{1}{8}\log C\right\} - \frac{8}{(C+1)\log t_0} \prod_{p\leqslant t_0} \left(1-\frac{1}{p}\right)^{-1}$$
$$\geqslant \exp\left\{-\frac{1}{8C\log t_0} - \frac{1}{4} - \frac{1}{8}\log C\right\} - \frac{8e^{\gamma}(1+1/\log^2 t_0)}{C+1}$$
$$\geqslant \exp\left\{-\frac{1}{256C} - \frac{1}{4} - \frac{1}{8}\log C\right\} - \frac{8e^{\gamma}(1+1/1024)}{C+1}.$$

Taking $C = 296$ produces a lower bound for the above product of 0.33437. Therefore,

$$\prod_{p\in Q} \left(1-\frac{1}{p}\right) \geqslant \frac{1}{16.016e^{\gamma}} \left(1-\frac{1}{t_0}\right) 0.33437 \geqslant \frac{1}{85.32}$$

and the proof of Theorem 4 is complete. □

Proof of Theorem 2. By Theorem 4,

$$\frac{n}{f(n)} = \prod_{p^a\|n} \frac{1}{1-p^{-a}} \leqslant \prod_{p\in Q} \frac{1}{1-p^{-1}} \prod_{p} \frac{1}{1-p^{-40}} \leqslant 85.4.$$ □

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DEPARTMENT OF MATHEMATICS, 1409 WEST GREEN STREET, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA

E-mail address: ford@math.uiuc.edu

STEKLOV MATHEMATICAL INSTITUTE, 8, GUBKIN STREET, MOSCOW, 119991, RUSSIA

E-mail address: konyagin@mi.ras.ru

CENTRO DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTONOMA DE MÉXICO, C.P. 58089, MORELIA, MICHOACÁN, MÉXICO

E-mail address: fluca@matmor.unam.mx