

# INTEGERS WITH A DIVISOR IN $(y, 2y]$

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ABSTRACT. We determine, up to multiplicative constants, how many integers  $n \leq x$  have a divisor in  $(y, 2y]$ .

## 1. INTRODUCTION

Let  $H(x, y, z)$  be the number of integers  $n \leq x$  which have a divisor in the interval  $(y, z]$ . In the author's paper [3], the correct order of growth of  $H(x, y, z)$  was determined for all  $x, y, z$ . In particular,

$$(1.1) \quad H(x, y, 2y) \asymp \frac{x}{(\log y)^\delta (\log \log y)^{3/2}} \quad (3 \leq y \leq \sqrt{x}),$$

where

$$\delta = 1 - \frac{1 + \log \log 2}{\log 2} = 0.086071 \dots$$

In this note we prove only the important special case (1.1), omitting the parts of the argument required for other cases. In addition, we present an alternate proof, dating from 2002, of the lower bound implicit in (1.1). This proof avoids the use of results about uniform order statistics required in [3], and instead utilizes the cycle lemma from combinatorics. Although shorter and technically simpler than the argument in [3], this method is not useful for counting integers with a prescribed number of divisors in  $(y, 2y]$ .

We mention here one of the applications of (1.1), a 50-year old problem of Erdős ([1], [2]) known colloquially as the “multiplication table problem”. Let  $A(x)$  be the number of positive integers  $n \leq x$  which can be written as  $n = m_1 m_2$  with each  $m_i \leq \sqrt{x}$ . Then

$$A(x) \asymp \frac{x}{(\log x)^\delta (\log \log x)^{3/2}}.$$

This follows directly from (1.1) and the inequalities

$$H\left(\frac{x}{4}, \frac{\sqrt{x}}{4}, \frac{\sqrt{x}}{2}\right) \leq A(x) \leq \sum_{k \geq 0} H\left(\frac{x}{2^k}, \frac{\sqrt{x}}{2^{k+1}}, \frac{\sqrt{x}}{2^k}\right).$$

More on the history of estimations of  $H(x, y, z)$ , further applications and references may be found in [3].

**Heuristic argument.** For brevity, let  $\tau(n, y, z)$  be the number of divisors of  $n$  in  $(y, z]$ . Write  $n = n'n''$ , where  $n'$  is composed only of primes  $\leq 2y$  and  $n''$  is composed only of primes

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*Date:* July 17, 2007.

2000 Mathematics Subject Classification: Primary 11N25; Secondary 62G30.

Research supported by National Science Foundation grants DMS-0301083 and DMS-0555367.

$> 2y$ . For simplicity, assume  $n'$  is squarefree and  $n' \leq y^{100}$ . Assume for the moment that the set  $D(n') = \{\log d : d|n'\}$  is uniformly distributed in  $[0, \log n']$ . If  $n'$  has  $k$  prime factors, then the expected value of  $\tau(n', y, 2y)$  should be about  $\frac{2^k \log 2}{\log n'} \asymp \frac{2^k}{\log y}$ . This is  $\gg 1$  precisely when  $k \geq k_0 + O(1)$ , where  $k_0 := \left\lfloor \frac{\log \log y}{\log 2} \right\rfloor$ . Using the fact (e.g. Theorem 08 of [5]) that the number of  $n \leq x$  with  $n'$  having  $k$  prime factors is of order

$$\frac{x}{\log y} \frac{(\log \log y)^k}{k!},$$

we obtain a heuristic estimate for  $H(x, y, 2y)$  of order

$$\frac{x}{\log y} \sum_{k \geq k_0 + O(1)} \frac{(\log \log y)^k}{k!} \asymp \frac{x(\log \log y)^{k_0}}{k_0! \log y} \asymp \frac{x}{(\log y)^\delta (\log \log y)^{1/2}}.$$

This is slightly too big, and the reason stems from the uniformity assumption about  $D(n')$ . In fact, for most  $n'$  with about  $k_0$  prime factors, the set  $D(n')$  is far from uniform, possessing many clusters of divisors and large gaps between clusters. This substantially decreases the likelihood that  $\tau(n', y, 2y) \geq 1$ . The numbers  $\log \log p$  over  $p|n'$  are well-known to behave like random numbers in  $[0, \log \log 2y]$ . Consequently, if we write  $n' = p_1 \cdots p_k$ , where  $p_1 < p_2 < \dots < p_k$ , then we expect  $\log \log p_j \approx \frac{j \log \log y}{k_0} = j \log 2 + O(1)$  for each  $j$ . Large deviation results from probability theory (see Smirnov's theorem in §4; also see Ch. 1 of [5]) tell us that with high probability there is a  $j$  for which  $\log \log p_j \leq j \log 2 - c\sqrt{\log \log y}$ , where  $c$  is a small positive constant. Thus, the  $2^j$  divisors of  $p_1 \cdots p_j$  will be clustered in an interval of logarithmic length about  $\ll \log p_j \leq 2^j e^{-c\sqrt{\log \log y}}$ . On a logarithmic scale, the divisors of  $n'$  will then lie in  $2^{k-j}$  translates of this cluster. A measure of the degree of clustering of the divisors of an integer  $a$  is given by

$$L(a) = \text{meas} \mathcal{L}(a), \quad \mathcal{L}(a) = \bigcup_{d|a} [-\log 2 + \log d, \log d].$$

The probability that  $\tau(n', y, 2y) \geq 1$  should then be about  $L(n')/\log y$ . Making this precise leads to the upper and lower bounds for  $H(x, y, 2y)$  given below in Lemmas 2.1 and 3.2. The upper bound for  $L(a)$  given in Lemma 3.1 (iii) below quantifies how small  $L(a)$  must be when there is a  $j$  with  $\log \log p_j$  considerably smaller than  $j \log 2$ .

What we really need to count is  $n$  for which  $n'$  has about  $k_0$  prime factors *and*  $L(n') \gg \log n'$ . This roughly corresponds to asking for  $\log \log p_j \geq j \log 2 - O(1)$  for all  $j$ . The analogous problem from statistics theory is to ask for the likelihood that given  $k_0$  random numbers in  $[0, 1]$ , there are  $\leq k_0 x + O(1)$  of them which are  $\leq x$ , uniformly in  $0 \leq x \leq 1$ . In section 4, Lemma 4.1, we will see that this probability is about  $1/k_0 \asymp 1/\log \log y$  and this leads to the correct order (1.1).

**Notation:** Let  $\tau(n)$  be the number of positive divisors of  $n$ , and define  $\omega(n)$  to be the number of distinct prime divisors of  $n$ . Let  $P^+(n)$  be the largest prime factor of  $n$  and let  $P^-(n)$  be the smallest prime factor of  $n$ . Adopt the notational conventions  $P^+(1) = 0$  and  $P^-(1) = \infty$ . Constants implied by  $O$ ,  $\ll$  and  $\asymp$  are absolute. The notation  $f \asymp g$  means  $f \ll g$  and  $g \ll f$ .

We shall make frequent use of the following estimate, which is a consequence of the Prime Number Theorem with classical de la Valée Poussin error term. For certain constants  $c_0, c_1$ ,

$$(1.2) \quad \sum_{p \leq x} \frac{1}{p} = \log \log x + c_0 + O(e^{-c_1 \sqrt{\log x}}) \quad (x \geq 2).$$

We also need the standard sieve bound (e.g. [4]; Theorem 06 and Exercise 02 of [5])

$$(1.3) \quad |\{n \leq x : P^-(n) > z\}| \asymp \frac{x}{\log z} \quad (x \geq 2z \geq 4)$$

and Stirling's formula  $k! \sim \sqrt{2\pi k}(k/e)^k$ .

## 2. LOWER BOUND

In this section we prove the lower bound implicit in (1.1). The first step is to bound  $H(x, y, 2y)$  in terms of a sum of  $L(a)/a$ . Next, sums of  $L(a)/a$  are related via the Cauchy-Schwarz inequality to sums of a function  $W(a)$  which counts pairs of divisors of  $a$  which are close together. With a strategic choice of sets of  $a$  to average over, the problem is reduced to the estimation of a certain combinatorial sum. This is accomplished with the aid of a tool closely related to the so-called ‘‘cycle lemma’’.

**Lemma 2.1.** *If  $3 \leq y \leq \sqrt{x}$ , then*

$$H(x, y, 2y) \gg \frac{x}{\log^2 y} \sum_{a \leq y^{1/8}} \frac{L(a)}{a}.$$

*Proof.* Let  $y_0$  be a sufficiently large constant. If  $3 \leq y \leq y_0$ , then  $H(x, y, 2y) \gg x \gg \frac{xL(1)}{\log^2 y}$  since  $L(1) = \log 2$ . If  $y \geq y_0$ , consider integers  $n = apb \leq x$  with  $a \leq y^{1/8}$ , all prime factors of  $b$  are  $> 2y$  or in  $[y^{1/4}, y^{3/4}]$ , and  $p$  is a prime with  $\log(y/p) \in \mathcal{L}(a)$ . The last condition implies that  $\tau(ap, y, 2y) \geq 1$ . In particular,  $y^{7/8} \leq y/a < p \leq 2y$ . Thus, each  $n$  has a unique representation in this form. Fix  $a$  and  $p$  and note that  $x/(ap) \geq x/(2y^{9/8}) \geq \frac{1}{2}y^{7/8}$ . If  $x/(ap) \geq 4y$ , (1.3) implies that the number of  $b \leq \frac{x}{ap}$  with  $P^-(b) > 2y$  is  $\gg \frac{x}{ap \log y}$ . If  $x/(ap) < 4y$ , then the number of  $b \leq \frac{x}{ap}$  composed of two prime factors in  $(y^{1/4}, y^{3/4}]$  is likewise  $\gg \frac{x}{ap \log y}$ . Hence

$$H(x, y, 2y) \gg \frac{x}{\log y} \sum_{a \leq y^{1/8}} \frac{1}{a} \sum_{\log(y/p) \in \mathcal{L}(a)} \frac{1}{p}$$

Since  $\mathcal{L}(a)$  is the disjoint union of intervals of length  $\geq \log 2$  and  $p \geq y^{7/8}$ , for each  $a$  we have by repeated application of (1.2)

$$\sum_{\log(y/p) \in \mathcal{L}(a)} \frac{1}{p} \gg \frac{L(a)}{\log y}.$$

□

**Lemma 2.2.** *For any set  $\mathcal{A}$  of positive integers,*

$$\sum_{a \in \mathcal{A}} \frac{L(a)}{a} \geq \frac{\left( \sum_{a \in \mathcal{A}} \frac{\tau(a)}{a} \right)^2}{6 \sum_{a \in \mathcal{A}} \frac{W(a)}{a}},$$

where

$$W(a) = |\{(d, d') : d|a, d'|a, |\log d/d'| \leq \log 2\}|.$$

*Proof.* Since  $\tau(a) \log 2 = \int \tau(a, e^u, 2e^u) du$ , by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left( \sum_{a \in \mathcal{A}} \frac{\tau(a)}{a} \right)^2 (\log 2)^2 &= \left( \sum_{a \in \mathcal{A}} \frac{1}{a} \int \tau(a, e^u, 2e^u) du \right)^2 \\ &\leq \left( \sum_{a \in \mathcal{A}} \frac{L(a)}{a} \right) \left( \sum_{a \in \mathcal{A}} \frac{1}{a} \int \tau^2(a, e^u, 2e^u) du \right). \end{aligned}$$

Partition the real numbers into intervals  $I_j$  of length  $\log 2$  and let  $k_j$  be the number of divisors of  $a$  in  $I_j$ . Then

$$\int \tau^2(a, e^u, 2e^u) du \leq (\log 2) \sum_j (k_j + k_{j+1})^2 \leq 4(\log 2) \sum_j k_j^2 \leq 4(\log 2)W(a).$$

□

We apply Lemma 2.2 with sets  $\mathcal{A}$  of integer having particular distribution of prime factors. Partition the primes into sets  $D_1, D_2, \dots$ , where each  $D_j$  consists of the primes in an interval  $(\lambda_{j-1}, \lambda_j]$ , with  $\lambda_j \approx \lambda_{j-1}^2$ . More precisely, let  $\lambda_0 = 1.9$  and inductively define  $\lambda_j$  for  $j \geq 1$  as the largest prime so that

$$(2.1) \quad \sum_{\lambda_{j-1} < p \leq \lambda_j} \frac{1}{p} \leq \log 2.$$

For example,  $\lambda_1 = 2$  and  $\lambda_2 = 7$ . By (1.2), we have

$$\log \log \lambda_j - \log \log \lambda_{j-1} = \log 2 + O(e^{-c_1} \sqrt{\log \lambda_{j-1}}),$$

and thus for some absolute constant  $K$ ,

$$(2.2) \quad 2^{j-K} \leq \log \lambda_j \leq 2^{j+K} \quad (j \geq 0).$$

For a vector  $\mathbf{b} = (b_1, \dots, b_J)$  of non-negative integers, let  $\mathcal{A}(\mathbf{b})$  be the set of square-free integers  $a$  composed of exactly  $b_j$  prime factors from  $D_j$  for each  $j$ .

**Lemma 2.3.** *Assume  $\mathbf{b} = (b_1, \dots, b_J)$ . Then*

$$\sum_{a \in \mathcal{A}(\mathbf{b})} \frac{W(a)}{a} \ll \frac{(2 \log 2)^{b_1 + \dots + b_J}}{b_1! \cdots b_J!} \sum_{j=1}^J 2^{-j + b_1 + \dots + b_j}.$$

*Proof.* Let  $B = b_1 + \cdots + b_J$  and for  $j \geq 0$  let  $b'_j = \sum_{i \leq j} b_j$ . Let  $a = p_1 \cdots p_B$ , where

$$(2.3) \quad pb'_{j-1+1}, \dots, pb'_j \in D_j \quad (1 \leq j \leq J)$$

and the primes in each interval  $D_j$  are unordered. Since  $W(p_1 \cdots p_B)$  is the number of pairs  $Y, Z \subseteq \{1, \dots, B\}$  with

$$(2.4) \quad \left| \sum_{i \in Y} \log p_i - \sum_{i \in Z} \log p_i \right| \leq \log 2,$$

we have

$$(2.5) \quad \sum_{a \in \mathcal{A}(\mathbf{b})} \frac{W(a)}{a} \leq \frac{1}{b_1! \cdots b_J!} \sum_{Y, Z \subseteq \{1, \dots, B\}} \sum_{\substack{p_1, \dots, p_B \\ (2.3), (2.4)}} \frac{1}{p_1 \cdots p_B}.$$

When  $Y = Z$ , (2.1) implies that the inner sum on the right side of (2.5) is  $\leq (\log 2)^B$ , and there are  $2^B$  such pairs  $Y, Z$ . When  $Y \neq Z$ , let  $I = \max[(Y \cup Z) - (Y \cap Z)]$ . With all the  $p_i$  fixed except for  $p_I$ , (2.4) implies that  $U \leq p_I \leq 4U$  for some number  $U$ . Let  $E(I)$  be defined by  $b'_{E(I)-1} < I \leq b'_{E(I)}$ , i.e.  $p_I \in D_{E(I)}$ . By (1.2),

$$\sum_{\substack{U \leq p_I \leq 4U \\ p_I \in D_{E(I)}}} \frac{1}{p_I} \ll \frac{1}{\max(\log U, \log \lambda_{E(I)-1})} \ll 2^{-E(I)}.$$

Thus, by (2.1) the inner sum in (2.5) is  $\ll 2^{-E(I)}(\log 2)^B$ . With  $I$  fixed, there correspond  $2^{B-I+1}4^{I-1} = 2^{B+I-1}$  pairs  $Y, Z$ . By (2.5),

$$\sum_{a \in \mathcal{A}(\mathbf{b})} \frac{W(a)}{a} \ll \frac{(2 \log 2)^B}{b_1! \cdots b_J!} \left[ 1 + \sum_{I=1}^B 2^{I-E(I)} \right] \ll \frac{(2 \log 2)^B}{b_1! \cdots b_J!} \sum_{j=1}^J 2^{-j} \sum_{b'_{j-1} < I \leq b'_j} 2^I,$$

and the claimed bound follows.  $\square$

Now suppose that  $M$  is a sufficiently large positive integer,  $b_i = 0$  for  $i < M$ , and  $b_j \leq Mj$  for each  $j$ . By (2.2),

$$(2.6) \quad \begin{aligned} \sum_{a \in \mathcal{A}(\mathbf{b})} \frac{\tau(a)}{a} &= 2^k \prod_{j=M}^J \frac{1}{b_j!} \left( \sum_{p_1 \in D_j} \frac{1}{p_1} \sum_{\substack{p_2 \in D_j \\ p_2 \neq p_1}} \frac{1}{p_2} \cdots \sum_{\substack{p_{b_j} \in D_j \\ p_{b_j} \notin \{p_1, \dots, p_{b_j-1}\}}} \frac{1}{p_{b_j}} \right) \\ &\geq 2^k \prod_{j=M}^J \frac{1}{b_j!} \left( \log 2 - \frac{b_j}{\lambda_{j-1}} \right)^{b_j} \\ &\geq \frac{(2 \log 2)^k}{2b_M! \cdots b_J!}. \end{aligned}$$

Let

$$k = \left\lfloor \frac{\log \log y}{\log 2} - 2M \right\rfloor, \quad J = M + k - 1.$$

Let  $\mathcal{B}$  be the set of vectors  $(b_1, \dots, b_J)$  with  $b_i = 0$  for  $i < M$  and  $b_1 + \dots + b_J = k$ . Let  $\mathcal{B}^*$  be the set of  $\mathbf{b} \in \mathcal{B}$  with  $b_j \leq \min(Mj, M(J-j+1))$  for each  $j \geq M$ . If  $\mathbf{b} \in \mathcal{B}^*$  and  $a \in \mathcal{A}(\mathbf{b})$ , then by (2.2),

$$\log a \leq \sum_{j=M}^J b_j \log \lambda_j \leq M \sum_{l=0}^{J-M} (l+1) 2^{J+K-l} < \frac{\log y}{8}$$

if  $M$  is large enough, as  $2^{J+1} \leq 2^{-M} \log y$ . Put

$$(2.7) \quad f(\mathbf{b}) = \sum_{h=M}^J 2^{M-1-h+b_M+\dots+b_h}.$$

By Lemma 2.3,

$$\sum_{a \in \mathcal{A}(\mathbf{b})} \frac{W(a)}{a} \ll \frac{(2 \log 2)^k}{b_M! \dots b_J!} (1 + 2^{1-M} f(\mathbf{b})) \ll \frac{(2 \log 2)^k}{b_M! \dots b_J!} f(\mathbf{b})$$

since  $f(\mathbf{b}) \geq 1/2$ . By Lemmas 2.1 and 2.2, plus (2.6), we have for large  $y$

$$(2.8) \quad H(x, y, 2y) \gg \frac{x(2 \log 2)^k}{\log^2 y} \sum_{\mathbf{b} \in \mathcal{B}^*} \frac{1}{b_M! \dots b_J! f(\mathbf{b})}.$$

Observe that the product of factorials is unchanged under permutation of  $b_M, \dots, b_J$ . Roughly speaking,

$$f(\mathbf{b}) \approx g(\mathbf{b}) := \max_j 2^{(b_M-1)+\dots+(b_j-1)}.$$

Note that  $(b_M - 1) + \dots + (b_J - 1) = k - (J - M + 1) = 0$ .

Given real numbers  $z_1, \dots, z_k$  with zero sum, there is a cyclic permutation  $\mathbf{z}'$  of the vector  $\mathbf{z} = (z_1, \dots, z_k)$  all of whose partial sums are  $\geq 0$ : let  $i$  be the index minimizing  $z_1 + \dots + z_i$  and take  $\mathbf{z}' = (z_i, \dots, z_k, z_1, \dots, z_{i-1})$ . In combinatorics, this fact is known as the *cycle lemma*. Thus, there is a cyclic permutation  $\mathbf{b}'$  of  $\mathbf{b}$  with  $g(\mathbf{b}') = 1$ . Thus, we expect that  $1/f(\mathbf{b}')$  will be  $\gg 1/k$  on average over  $\mathbf{b}'$  and that  $1/f(\mathbf{b}) \gg 1/k$  on average over  $\mathbf{b} \in \mathcal{B}$ . This is essentially what we prove next; see (2.10) below.

**Lemma 2.4.** *For positive real numbers  $x_1, \dots, x_r$  with product  $X$ , let  $x_{r+i} = x_i$  for  $i \geq 1$ . Then*

$$\sum_{j=0}^{r-1} \left( \sum_{h=1}^r x_{1+h} \dots x_{h+j} \right)^{-1} \in \left[ \frac{1}{\max(1, X)}, \frac{1}{\min(1, X)} \right].$$

*Proof.* Put  $y_0 = 1$  and  $y_j = x_1 \dots x_j$  for  $j \geq 1$ . The sum in question is

$$\sum_{j=0}^{r-1} \left( \sum_{h=1}^r \frac{y_{h+j}}{y_j} \right)^{-1} = \sum_{j=0}^{r-1} \frac{y_j}{y_{1+j} + \dots + y_{r+j}}.$$

Since  $y_r = X$ ,

$$\begin{aligned} y_{1+j} + \dots + y_{r+j} &= X(y_0 + \dots + y_j) + y_{1+j} + \dots + y_{r-1} \\ &\in [\min(1, X)(y_0 + \dots + y_{r-1}), \max(1, X)(y_0 + \dots + y_{r-1})]. \end{aligned}$$

□

We have

$$(2.9) \quad \sum_{\mathbf{b} \in \mathcal{B}^*} \frac{1}{b_M! \cdots b_J! f(\mathbf{b})} \geq S_0 - \sum_{M \leq j < k/M} S_1(j) - \sum_{1 \leq j < k/M} S_2(j),$$

where

$$\begin{aligned} S_0 &= \sum_{\mathbf{b} \in \mathcal{B}} \frac{1}{b_M! \cdots b_J! f(\mathbf{b})}, \\ S_1(j) &= \sum_{\substack{\mathbf{b} \in \mathcal{B} \\ b_j > Mj}} \frac{1}{b_M! \cdots b_J! f(\mathbf{b})}, \\ S_2(j) &= \sum_{\substack{\mathbf{b} \in \mathcal{B} \\ b_{J+1-j} > Mj}} \frac{1}{b_M! \cdots b_J! f(\mathbf{b})}. \end{aligned}$$

Let  $x_i = 2^{-1+b_{M-1+i}}$  for  $1 \leq i \leq k$ . Then  $x_1 \cdots x_k = 1$  and

$$f(\mathbf{b}) = x_1 + x_1 x_2 + \cdots + x_1 x_2 \cdots x_k.$$

By Lemma 2.4 and the multinomial theorem,

$$(2.10) \quad S_0 = \sum_{\mathbf{b} \in \mathcal{B}} \frac{1}{b_M! \cdots b_J!} \frac{1}{k} \sum_{j=0}^{k-1} \left( \sum_{h=1}^k x_{1+j} \cdots x_{h+j} \right)^{-1} = \frac{k^{k-1}}{k!}.$$

To bound  $S_1(j)$ , apply Lemma 2.4 with  $x_i = 2^{b_{j+i}-1}$  for  $1 \leq i \leq J-j$  and note that

$$X = x_1 \cdots x_{J-j} = 2^{j+1-M-b_M-\cdots-b_j} < 1.$$

Write  $\mathbf{b}' = (b_M, \dots, b_{j-1}, b_{j+1}, \dots, b_J)$ , whose sum of components is  $k - b_j$ . Ignoring the terms with  $h \leq j$  in (2.7), using Lemma 2.4 and the multinomial theorem, we find

$$\begin{aligned} S_1(j) &\leq \sum_{b_j > Mj} \frac{1}{b_j!} \sum_{\mathbf{b}'} \frac{1}{\prod_{i \neq j} b_i!} \frac{1}{2^{M-1-j+b_M+\cdots+b_j}} \frac{1}{J-j} \sum_{i=0}^{J-j-1} \left( \sum_{h=1}^{J-j} x_{1+i} \cdots x_{h+i} \right)^{-1} \\ &\leq \frac{1}{J-j} \sum_{b_j > Mj} \frac{(k-1)^{k-b_j}}{b_j! (k-b_j)!} \\ &\leq \frac{2k^{k-1}}{k!} \sum_{b_j > Mj} \frac{1}{b_j!} \\ &\leq \frac{k^{k-1}}{k!} \frac{2}{(Mj)!}. \end{aligned}$$

Hence, if  $M \geq 2$  then

$$(2.11) \quad \sum_{M \leq j < k/M} S_1(j) \leq \frac{k^{k-1}}{10k!}.$$

The estimation of  $S_2(j)$  is similar. Let  $x_i = 2^{-1+b_{M+i-1}}$  for  $1 \leq i \leq J - M + 1 - j$ , so that

$$X = x_1 \cdots x_{J-M+1-j} = 2^{j-b_{J-j+1}-\cdots-b_J} \leq 1.$$

Put  $b = b_{J-j+1}$  and let  $\mathbf{b}' = (b_M, \dots, b_{J-j}, b_{J-j+2}, \dots, b_J)$ , whose sum of components is  $k - b$ . Then, ignoring the terms with  $h > J - j$  in (2.7), we have

$$\begin{aligned} S_2(j) &\leq \sum_{b > M_j} \frac{1}{b!} \sum_{\mathbf{b}'} \frac{1}{\prod_{i \neq J-j+1} b_i!} \frac{2^{b-j+b_{J-j+2}+\cdots+b_J}}{J-M+1-j} \\ &= \frac{2^{-j}}{J-M+1-j} \sum_{b > M_j} \frac{2^b}{b!} \frac{(k+j-2)^{k-b}}{(k-b)!} \\ &\leq \frac{2^{1-j}}{k \cdot k!} (k+j)^k \sum_{b > M_j} \frac{2^b}{b!} \\ &\leq \frac{k^{k-1}}{k!} 2^{1-j} e^j \frac{2^{M_j}}{(M_j)!}. \end{aligned}$$

If  $M$  is large enough, then

$$(2.12) \quad \sum_{j \geq 1} S_2(j) \leq \frac{k^{k-1}}{10k!}.$$

By (2.9), (2.10), (2.11) and (2.12),

$$\sum_{\mathbf{b} \in \mathcal{B}^*} \frac{1}{b_M! \cdots b_J! f(\mathbf{b})} \geq \frac{k^{k-1}}{2k!}.$$

The lower bound in (1.1) for large  $y$  now follows from (2.8) and Stirling's formula. If  $y \leq y_0$  for some fixed constant  $y_0$ , the lower bound in (1.1) follows from  $H(x, y, 2y) \gg x$ .

### 3. UPPER BOUND, PART I

In this section, we prove the upper bound implicit in (1.1), except for the estimation of some integrals which will be dealt with in section 4. As with the lower bound argument, we begin by bounding  $H(x, y, 2y)$  in terms of a sum involving  $L(a)$ . Using a relatively simple upper bound for  $L(a)$  proved in Lemma 3.1 below, the sums involving  $L(a)$  are bounded in terms of particular multivariate integrals. The estimates for these integrals in section 4 allow us then to complete the proof.

**Lemma 3.1.** *We have*

- (i)  $L(a) \leq \min(\tau(a) \log 2, \log 2 + \log a)$ ;
- (ii) *If*  $(a, b) = 1$ , *then*  $L(ab) \leq \tau(b)L(a)$ ;
- (iii) *If*  $p_1 < \cdots < p_k$ , *then*

$$L(p_1 \cdots p_k) \leq \min_{0 \leq j \leq k} 2^{k-j} (\log(p_1 \cdots p_j) + \log 2).$$



*Proof.* Part (i) is immediate, since  $\mathcal{L}(a)$  is the union of  $\tau(a)$  intervals of length  $\log 2$ , all contained in  $[-\log 2, \log a)$ . Part (ii) follows from

$$\mathcal{L}(ab) = \bigcup_{d|b} \{u + \log d : u \in \mathcal{L}(a)\}.$$

Combining parts (i) and (ii) with  $a = p_1 \cdots p_j$  and  $b = p_{j+1} \cdots p_k$  yields (iii).  $\square$

**Remarks.** More generally, we may define for non-negative real numbers  $x_1, \dots, x_k$  the quantity

$$\mathcal{L}(\mathbf{x}; \eta) = \bigcup_{\varepsilon_1, \dots, \varepsilon_k \in \{0,1\}} \left( -\eta + \sum_{i=1}^k \varepsilon_i x_i, \sum_{i=1}^k \varepsilon_i x_i \right).$$

Then  $L(\mathbf{x}; \eta) = \text{meas} \mathcal{L}(\mathbf{x}; \eta)$  measures the distribution of the subset sums of  $x_1, \dots, x_k$ . The proof of Lemma 3.1 (iii) gives

$$\mathcal{L}(\mathbf{x}; \eta) \leq \min_{0 \leq j \leq k} 2^{k-j} (x_1 + \cdots + x_j + \eta).$$

**Lemma 3.2.** *If  $3 \leq y \leq \sqrt{x}$ , then*

$$H(x, y, 2y) \ll x \max_{\sqrt{y} \leq t \leq x} S(t),$$

where

$$(3.1) \quad S(t) = \sum_{\substack{P^+(a) \leq t \\ \mu^2(a)=1}} \frac{L(a)}{a \log^2(t/a + P^+(a))}.$$

*Proof.* First, we reduce the problem to estimating  $H^*(x, y, z)$ , the number of *squarefree* integers  $n \leq x$  with  $\tau(n, y, z) \geq 1$ . Write  $n = n'n''$ , where  $n'$  is squarefree,  $n''$  is squarefull and  $(n', n'') = 1$ . The number of  $n \leq x$  with  $n'' > (\log y)^4$  is

$$\leq x \sum_{n'' > (\log y)^4} \frac{1}{n''} \ll \frac{x}{(\log y)^2}.$$

Assume now that  $n'' \leq (\log y)^4$ . For some  $f|n''$ ,  $n'$  has a divisor in  $(y/f, 2y/f]$ , hence

$$(3.2) \quad H(x, y, 2y) \leq \sum_{n'' \leq (\log y)^4} \sum_{f|n''} H^*\left(\frac{x}{n''}, \frac{y}{f}, \frac{2y}{f}\right) + O\left(\frac{x}{(\log y)^2}\right).$$

Next, we show that for  $3 \leq y_1 \leq x_1^{3/5}$ ,

$$(3.3) \quad H^*(x_1, y_1, 2y_1) - H^*\left(\frac{1}{2}x_1, y_1, 2y_1\right) \ll x_1 (S(2y_1) + S(x_1/y_1)).$$

Let  $\mathcal{A}$  be the set of squarefree integers  $n \in (\frac{1}{2}x_1, x_1]$  with a divisor in  $(y_1, 2y_1]$ . Put  $z_1 = 2y_1$ ,  $y_2 = \frac{x_1}{4y_1}$ ,  $z_2 = \frac{x_1}{y_1}$ . If  $n \in \mathcal{A}$ , then  $n = m_1 m_2$  with  $y_i < m_i \leq z_i$  ( $i = 1, 2$ ). For some  $j \in \{1, 2\}$  we have  $p = P^+(m_j) < P^+(m_{3-j})$ . Write  $n = abp$ , where  $P^+(a) < p < P^-(b)$  and  $b > p$ . Since  $\tau(ap, y_j, z_j) \geq 1$ , we have  $p \geq y_j/a$ . By (1.3), given  $a$  and  $p$ , the number of possible  $b$  is

$$\ll \frac{x_1}{ap \log p} \leq \frac{x_1}{ap \log Q},$$

where  $Q = \max(P^+(a), y_j/a)$ . Since  $a$  has a divisor in  $(y_j/p, z_j/p]$ , we have  $\log(y_j/p) \in \mathcal{L}(a)$  or  $\log(2y_j/p) \in \mathcal{L}(a)$ . Since  $\mathcal{L}(a)$  is the disjoint union of intervals of length  $\geq \log 2$  with total measure  $L(a)$ , by repeated use of (1.2) we obtain

$$\sum_{\substack{\log(cy_j/p) \in \mathcal{L}(a) \\ p \geq Q}} \frac{1}{p} \ll \frac{L(a)}{\log Q} \quad (c = 1, 2),$$

and (3.3) follows.

Write  $x_2 = x/n''$ ,  $y_1 = y/f$ . Each  $n \in (x_2/\log^2 y_1, x_2]$  lies in an interval  $(2^{-r+1}x_2, 2^{-r}x_2]$  for some integer  $0 \leq r \leq 5 \log \log y_1$ . Applying (3.3) with  $x_1 = 2^{-r}x_2$  for each  $r$  gives

$$H^*(x_2, y_1, 2y_1) \ll \frac{x_2}{\log^2 y_1} + \sum_r 2^{-r}x_2 (S(2y_1) + S(2^{-r}x_2/y_1)) \ll x_2 \max_{\sqrt{y} \leq t \leq x_2} S(t).$$

Here we used the fact that  $S(t) \geq \frac{L(1)}{\log^2 t} = \frac{\log 2}{\log^2 t}$ . Finally,  $\sum_{n''} \tau(n'')/n'' = O(1)$  and the lemma follows from (3.2).  $\square$

Define

$$(3.4) \quad T(P, Q) = \sum_{\substack{P^+(a) \leq P \\ a \geq Q, \mu^2(a)=1}} \frac{L(a)}{a}.$$

If  $a \leq t^{1/2}$  or  $P^+(a) > t^{1/3000}$ , then  $\log^2(t/a + P^+(a)) \gg \log^2 t$ . Otherwise,  $e^{e^{g-1}} < P^+(a) \leq e^{e^g}$  for some integer  $g$  satisfying  $0 \leq g \leq \log \log t - 5$ . Thus we have

$$(3.5) \quad S(t) \ll \frac{T(t, 1)}{\log^2 t} + \sum_{g_0 \leq g \leq \log \log t - 5} e^{-2g} T(\exp\{e^g\}, t^{1/2})$$

for a fixed large integer  $g_0$ . Further define

$$T_k(P, Q) = \sum_{\substack{P^+(a) \leq P, a \geq Q \\ \omega(a)=k, \mu^2(a)=1}} \frac{L(a)}{a}.$$

We next bound  $T_k(P, Q)$  in terms of a multivariate integral. Heuristically,  $\sum_{p \leq z} 1/p \approx \log \log z$ , hence by partial summation we expect for ‘‘nice’’ functions  $f$  that

$$\sum_{p_1 < \dots < p_k \leq P} \frac{f\left(\frac{\log \log p_1}{\log \log P}, \dots, \frac{\log \log p_k}{\log \log P}\right)}{p_1 \cdots p_k} \approx (\log \log P)^k \int \cdots \int_{0 \leq \xi_1 \leq \dots \leq \xi_k \leq 1} f(\boldsymbol{\xi}) d\boldsymbol{\xi}.$$

**Lemma 3.3.** *Suppose  $P$  is large and  $Q \geq 1$ . Let  $v = \left\lfloor \frac{\log \log P}{\log 2} \right\rfloor$  and suppose  $1 \leq k \leq 10v$ . Then*

$$T_k(P, Q) \ll e^{-\frac{\log Q}{\log P}} (2 \log \log P)^k U_k(v),$$

where

$$U_k(v) = \int \cdots \int_{0 \leq \xi_1 \leq \dots \leq \xi_k \leq 1} \min_{0 \leq j \leq k} 2^{-j} (2^{v\xi_1} + \cdots + 2^{v\xi_j} + 1) d\boldsymbol{\xi}.$$

*Proof.* Let  $\alpha = \frac{1}{\log P}$ . We have

$$T_k(P, Q) \leq Q^{-\alpha} \sum_{\substack{P^+(a) \leq P \\ \omega(a)=k}} \frac{L(a)}{a^{1-\alpha}}.$$

Also

$$(3.6) \quad \sum_{p \leq P} \frac{1}{p^{1-\alpha}} = \sum_{p \leq P} \frac{1}{p} + O\left(\alpha \sum_{p \leq P} \frac{\log p}{p}\right) = \log \log P + O(1).$$

In a similar manner to how we constructed the sets  $D_j$  in section 2, we find that there is an absolute constants  $K$  so that the following holds for all  $P$ : the interval  $[2, P]$  may be partitioned into subintervals  $E_0, \dots, E_{v+K-1}$  with  $v = \left\lfloor \frac{\log \log P}{\log 2} \right\rfloor$  and for each  $j$ ,

$$\sum_{p \in E_j} \frac{1}{p^{1-1/\log P}} \leq \log 2$$

and

$$(3.7) \quad p \in E_j \implies \log \log p \leq (j + K) \log 2.$$

Consider  $a = p_1 \cdots p_k$ ,  $p_1 < \cdots < p_k \leq P$  and define  $j_i$  by  $p_i \in E_{j_i}$  ( $1 \leq i \leq k$ ). Put  $l_i = \frac{\log \log p_i}{\log 2}$ . By Lemma 3.1 (iii) and (3.7),

$$L(a) \leq 2^k \min_{0 \leq g \leq k} 2^{-g} (2^{l_1} + \cdots + 2^{l_g} + 1) \leq 2^{k+K} F(\mathbf{j}),$$

where

$$F(\mathbf{j}) = \min_{0 \leq g \leq k} 2^{-g} (2^{j_1} + \cdots + 2^{j_g} + 1).$$

Let  $J$  denote the set of vectors  $\mathbf{j}$  satisfying  $0 \leq j_1 \leq \cdots \leq j_k \leq v + K - 1$ . Then

$$T_k(P, Q) \leq Q^{-\alpha} 2^{k+K} \sum_{\mathbf{j} \in J} F(\mathbf{j}) \sum_{\substack{p_1 < \cdots < p_k \\ p_i \in E_{j_i} (1 \leq i \leq k)}} \frac{1}{(p_1 \cdots p_k)^{1-\alpha}}.$$

Let  $b_j$  be the number of primes  $p_i$  in  $E_j$  for  $0 \leq j \leq v + K - 1$ . Using the hypothesis that  $k \leq 10v$ , the sum over  $p_1, \dots, p_k$  above is at most

$$\begin{aligned} \prod_{j=0}^{v+K-1} \frac{1}{b_j!} \left( \sum_{p \in E_j} \frac{1}{p^{1-\alpha}} \right)^{b_j} &\leq \frac{(\log 2)^k}{b_0! \cdots b_{v+K-1}!} \\ &= ((v+K) \log 2)^k \int_{R(\mathbf{j})} 1 d\boldsymbol{\xi} \\ &\leq e^{10K} (v \log 2)^k \int_{R(\mathbf{j})} 1 d\boldsymbol{\xi}, \end{aligned}$$

where

$$R(\mathbf{j}) = \{0 \leq \xi_1 \leq \cdots \leq \xi_k \leq 1 : j_i \leq (v+K)\xi_i \leq j_i + 1 \ \forall i\}.$$

Observe that, in  $R(\mathbf{j})$ , there are  $b_s$  numbers  $\xi_j$  satisfying  $s \leq (v+K)\xi_i \leq s+1$  for each  $s$ , and  $\text{Vol}\{0 \leq x_1 \leq \dots \leq x_b \leq 1\} = 1/b!$ . Since  $2^{j_i} \leq 2^{(v+K)\xi_i} \leq 2^K 2^{v\xi_i}$  for  $\boldsymbol{\xi} \in R(\mathbf{j})$ , we have

$$F(\mathbf{j}) \leq 2^K \min_{0 \leq g \leq k} 2^{-g} (2^{v\xi_1} + \dots + 2^{v\xi_g} + 1).$$

Hence

$$\sum_{\mathbf{j} \in J} F(\mathbf{j}) \int_{R(\mathbf{j})} 1 d\boldsymbol{\xi} \leq 2^K U_k(v)$$

and the lemma follows.  $\square$

Estimating  $U_k(v)$  is the most complex part of the argument. The next lemma will be proved in section 4.

**Lemma 3.4.** *Suppose  $k, v$  are integers with  $0 \leq k \leq 10v$ . Then*

$$U_k(v) \ll \frac{1 + |v - k|^2}{(k+1)!(2^{k-v} + 1)}.$$

Notice that the bound in Lemma 3.4 undergoes a change of behavior at  $k = v$ .

**Lemma 3.5.** *Suppose  $P$  is large and  $Q \geq 1$ . Then*

$$T(P, Q) \ll e^{-\frac{\log Q}{\log P}} \frac{(\log P)^{2-\delta}}{(\log \log P)^{3/2}}.$$

*Proof.* Let  $v = \left\lfloor \frac{\log \log P}{\log 2} \right\rfloor$  and  $\gamma = e^{-\frac{\log Q}{\log P}}$ . By Lemmas 3.3 and 3.4,

$$(3.8) \quad \sum_{v \leq k \leq 10v} T_k(P, Q) \ll \gamma \sum_{v \leq k \leq 10v} \frac{(k-v)^2 + 1}{2^{k-v}} \frac{(2 \log \log P)^k}{(k+1)!} \ll \frac{\gamma (2 \log \log P)^v}{(v+1)!}$$

and

$$(3.9) \quad \sum_{1 \leq k \leq v} T_k(P, Q) \ll \gamma \sum_{1 \leq k \leq v} \frac{((v-k)^2 + 1)(2 \log \log P)^k}{(k+1)!} \ll \gamma \frac{(2 \log \log P)^v}{(v+1)!}.$$

By Lemma 3.1 (i),

$$\begin{aligned} \sum_{k \geq 10v} T_k(P, Q) &\leq \sum_{k \geq 10v} \sum_{\substack{P^+(a) \leq P \\ \mu^2(a)=1, \omega(a)=k \\ a \geq Q}} \frac{2^k \log 2}{a} \\ &\leq \sum_{k \geq 10v} 2^k Q^{-1/\log P} \sum_{\substack{P^+(a) \leq P \\ \mu^2(a)=1, \omega(a)=k}} \frac{1}{a^{1-1/\log P}} \\ &\leq \gamma \sum_{k \geq 10v} \frac{2^k}{k!} \left( \sum_{p \leq P} \frac{1}{p^{1-1/\log P}} \right)^k. \end{aligned}$$

By (3.6), the sum on  $p$  is  $\log \log P + O(1)$ . Thus,

$$(3.10) \quad \sum_{k \geq 10v} T_k(P, Q) \leq \gamma \frac{(2 \log \log P + O(1))^{10v}}{(10v)!} \ll \gamma \frac{(2 \log \log P)^v}{(v+1)!}.$$

Finally,  $T_0(P, Q) = 0$  if  $Q > 1$  and  $T_0(P, 1) = L(1) = \log 2$ . Recalling the definition of  $v$  and combining (3.8), (3.9), (3.10) with Stirling's formula completes the proof.  $\square$

By Lemma 3.5 and (3.5),

$$\begin{aligned} S(t) &\ll \frac{(\log t)^{2-\delta}}{(\log t)^2 (\log \log t)^{3/2}} + \sum_{g_0 \leq g \leq \log \log t - 5} e^{-2g - (\log t)/(2e^g) + g(2-\delta)} g^{-3/2} \\ &\ll \frac{1}{(\log t)^\delta (\log \log t)^{3/2}}. \end{aligned}$$

The desired upper bound for  $H(x, y, 2y)$  now follows from Lemma 3.2.

#### 4. UPPER BOUND, PART II

The goal of this section is to prove Lemma 3.4, and thus complete the proof of the upper bound in (1.1).

Let  $Y_1, \dots, Y_n$  be independent, uniformly distributed random variables in  $[0, 1]$ . Let  $\xi_1$  be the smallest of the numbers  $Y_i$ , let  $\xi_2$  be the next smallest, etc., so that  $0 \leq \xi_1 \leq \dots \leq \xi_n \leq 1$ . The numbers  $\xi_i$  are the *order statistics* for  $Y_1, \dots, Y_n$ . Then  $k!U_k(v)$  is the expectation of the random variable

$$X = \min_{0 \leq j \leq k} 2^{-j} (2^{v\xi_1} + \dots + 2^{v\xi_j} + 1).$$

Heuristically, we expect that

$$(4.1) \quad \mathbb{E}X \ll \mathbb{E} \min_{1 \leq j \leq k} 2^{-j+v\xi_j},$$

so we need to understand the distribution of  $\min_{1 \leq j \leq k} v\xi_j - j$ . Let  $Q_k(u, v)$  be the probability that  $\xi_i \geq \frac{i-u}{v}$  for every  $i$ . In the special case  $v = k$ , Smirnov in 1939 showed that

$$Q_k(x\sqrt{k}, k) \sim 1 - e^{-2x^2}$$

for each fixed  $x$ . The corresponding probability estimate for two-sided bounds on the  $\xi_i$  was established by Kolmogorov in 1933 and together these limit theorems are the basis of the *Kolmogorov-Smirnov goodness-of-fit statistical tests*.

In the next lemma, we prove new, uniform estimates for  $Q_k(u, v)$ . The remainder of the section is essentially devoted to proving (4.1). The details are complicated, but the basic idea is that if  $2^{-j}(2^{v\xi_1} + \dots + 2^{v\xi_j})$  is much large than  $2^{v\xi_j - j}$ , then for some large  $l$ , the numbers  $\xi_{j-l}, \dots, \xi_j$  are all very close to one another. As shown below in Lemmas 4.3 and 4.4, this is quite rare.

**Lemma 4.1.** *Let  $w = u + v - k$ . Uniformly in  $u \geq 0$  and  $w \geq 0$ , we have*

$$Q_k(u, v) \ll \frac{(u+1)(w+1)^2}{k}.$$

*Proof.* Without loss of generality, suppose  $k \geq 100$ ,  $u \leq k/10$  and  $w \leq \sqrt{k}$ . If  $\min_{1 \leq i \leq k} (\xi_i - \frac{i-u}{v}) < 0$ , let  $l$  be the smallest index with  $\xi_l < \frac{l-u}{v}$  and write  $\xi_l = \frac{l-u-\lambda}{v}$ , so that  $0 \leq \lambda \leq 1$ . Let

$$R_l(\lambda) = \text{Vol} \left\{ 0 \leq \xi_1 \leq \cdots \leq \xi_{l-1} \leq \frac{l-u-\lambda}{v} : \xi_i \geq \frac{i-u}{v} (1 \leq i \leq l-1) \right\}.$$

Then we have

$$\begin{aligned} Q_k(u, v) &= 1 - \frac{k!}{v} \int_0^1 \sum_{u+\lambda \leq l \leq k} R_l(\lambda) \text{Vol} \left\{ \frac{l-u-\lambda}{v} \leq \xi_{l+1} \leq \cdots \leq \xi_k \leq 1 \right\} d\lambda \\ &= 1 - \frac{k!}{v} \int_0^1 \sum_{u+\lambda \leq l \leq k} \frac{R_l(\lambda)}{(k-l)!} \left( \frac{k+w+\lambda-l}{v} \right)^{k-l} d\lambda. \end{aligned}$$

Now suppose that  $\xi_k \leq 1 - \frac{2w+2}{v} = \frac{k-u-w-2}{v}$ . Then  $\min_{1 \leq i \leq k} \xi_i - \frac{i-u}{v} < 0$ . Defining  $l$  and  $\lambda$  as before, we have

$$\begin{aligned} \left( 1 - \frac{2w+2}{v} \right)^k &= k! \text{Vol} \left\{ 0 \leq \xi_1 \leq \cdots \leq \xi_k \leq 1 - \frac{2w+2}{v} \right\} \\ &= \frac{k!}{v} \int_0^1 \sum_{u+\lambda \leq l \leq k-w-2+\lambda} \frac{R_l(\lambda)}{(k-l)!} \left( \frac{k-l-w-2+\lambda}{v} \right)^{k-l} d\lambda. \end{aligned}$$

Thus, for any  $A > 0$ , we have

$$\begin{aligned} Q_k(u, v) &= 1 - A \left( 1 - \frac{2w+2}{v} \right)^k - \frac{k!}{v} \int_0^1 \sum_{k-w-2+\lambda < l \leq k} \frac{R_l(\lambda)}{(k-l)!} \left( \frac{k+w+\lambda-l}{v} \right)^{k-l} d\lambda \\ &\quad + \frac{k!}{v} \int_0^1 \sum_{u+\lambda \leq l \leq k-w-2+\lambda} \frac{R_l(\lambda)}{(k-l)! v^{k-l}} [A(k-l-w-2+\lambda)^{k-l} - (k-l+w+\lambda)^{k-l}] d\lambda. \end{aligned}$$

Noting that  $2 - \lambda \geq \lambda$ , we have

$$\begin{aligned} \left( \frac{k-l-w-2+\lambda}{k-l+w+\lambda} \right)^{k-l} &= \left( 1 - \frac{w+2-\lambda}{k-l} \right)^{k-l} \left( 1 + \frac{w+\lambda}{k-l} \right)^{-(k-l)} \\ &= \exp \left\{ -(2w+2) + \sum_{j=2}^{\infty} \frac{-(w+2-\lambda)^j + (-1)^j (w+\lambda)^j}{j(k-l)^{j-1}} \right\} \\ &\leq e^{-(2w+2)}. \end{aligned}$$

Thus, taking  $A = e^{2w+2}$ , we conclude that

$$\begin{aligned}
 Q_k(u, v) &\leq 1 - e^{2w+2} \left(1 - \frac{2w+2}{v}\right)^k \\
 &= 1 - \exp \left\{ \frac{2w+2}{v} (v - k + O(w)) \right\} \\
 &= 1 - \exp \left\{ \frac{-2uw + O(u + w^2 + 1)}{v} \right\} \\
 &\leq \frac{2uw + O(u + w^2 + 1)}{v} \ll \frac{(u+1)(w+1)^2}{k}.
 \end{aligned}$$

□

**Lemma 4.2.** *If  $t \geq 2$ ,  $b \geq 0$  and  $t + a + b > 0$ , then*

$$\sum_{\substack{1 \leq j \leq t-1 \\ j+a > 0}} \binom{t}{j} (a+j)^{j-1} (b+t-j)^{t-j-1} \leq e^4 (t+a+b)^{t-1}.$$

*Proof.* Let  $C_t(a, b)$  denote the sum in the lemma. We may assume that  $a > 1 - t$ , otherwise  $C_t(a, b) = 0$ . The associated “complete” sum is evaluated exactly using one of Abel’s identities ([6], p.20, equation (20))

$$(4.2) \quad \sum_{j=0}^t \binom{t}{j} (a+j)^{j-1} (b+t-j)^{t-j-1} = \left(\frac{1}{a} + \frac{1}{b}\right) (t+a+b)^{t-1} \quad (ab \neq 0).$$

If  $a \geq -1$ , put  $A = \max(1, a)$  and  $B = \max(1, b)$ . By (4.2),

$$\begin{aligned}
 (4.3) \quad C_t(a, b) &\leq C_t(A, B) \leq \left(\frac{1}{A} + \frac{1}{B}\right) (t+A+B)^{t-1} \\
 &\leq 2(t+a+b+3)^{t-1} \\
 &\leq 2e^{\frac{3(t-1)}{t+a+b}} (t+a+b)^{t-1} < e^4 (t+a+b)^{t-1}.
 \end{aligned}$$

Next assume  $a < -1$ . Since  $(1 + c/x)^x$  is an increasing function for  $x > 1$ , we have

$$(a+j)^{j-1} = (j-1)^{j-1} \left(1 + \frac{a+1}{j-1}\right)^{j-1} \leq (j-1)^{j-1} \left(1 + \frac{a+1}{t-1}\right)^{t-1}.$$

Thus, by (4.3),

$$\begin{aligned}
 C_t(a, b) &\leq \left(\frac{t+a}{t-1}\right)^{t-1} C_t(-1, b) \\
 &\leq e^4 \left(\frac{(t+a)(t+b-1)}{t-1}\right)^{t-1} = e^4 \left(t+a+b + \frac{(a+1)b}{t-1}\right)^{t-1} \\
 &\leq e^4 (t+a+b)^{t-1}.
 \end{aligned}$$

□

For brevity, write

$$S_k(u, v) = \{\boldsymbol{\xi} : 0 \leq \xi_1 \leq \cdots \leq \xi_k \leq 1 : \xi_i \geq \frac{i-u}{v} (1 \leq i \leq k)\},$$

so that  $Q_k(u, v) = k! \text{Vol } S_k(u, v)$ .

**Lemma 4.3.** *Suppose  $g, k, s, u, v \in \mathbb{Z}$  satisfy*

$$2 \leq g \leq k/2, \quad s \geq 0, \quad v \geq k/10, \quad u \geq 0, \quad u + v \geq k + 1.$$

Let  $R$  be the subset of  $\boldsymbol{\xi} \in S_k(u, v)$  where, for some  $l \geq g + 1$ , we have

$$(4.4) \quad \frac{l-u}{v} \leq \xi_l \leq \frac{l-u+1}{v}, \quad \xi_{l-g} \geq \frac{l-u-s}{v}.$$

Then

$$\text{Vol}(R) \ll \frac{(10(s+1))^g (u+1)(u+v-k)^2}{(g-2)! (k+1)!}.$$

*Proof.* Fix  $l$  satisfying  $\max(u, g+1) \leq l \leq k$ . Let  $R_l$  be the subset of  $\boldsymbol{\xi} \in S_k(u, v)$  satisfying (4.4) for this particular  $l$ . We have  $\text{Vol}(R_l) \leq V_1 V_2 V_3 V_4$ , where, by Lemma 4.1,

$$\begin{aligned} V_1 &= \text{Vol}\{0 \leq \xi_1 \leq \cdots \leq \xi_{l-g-1} \leq \frac{l-u+1}{v} : \xi_i \geq \frac{i-u}{v} \forall i\} \\ &= \left(\frac{l-u+1}{v}\right)^{l-g-1} \text{Vol}\{0 \leq \theta_1 \leq \cdots \leq \theta_{l-g-1} \leq 1 : \theta_i \geq \frac{i-u}{l-u+1} \forall i\} \\ &= \left(\frac{l-u+1}{v}\right)^{l-g-1} \frac{Q_{l-g-1}(u, l-u+1)}{(l-g-1)!} \\ &\ll \left(\frac{l-u+1}{v}\right)^{l-g-1} \frac{(u+1)g^2}{(l-g)!}, \end{aligned}$$

$$V_2 = \text{Vol}\left\{\frac{l-u-s}{v} \leq \xi_{l-g} \leq \cdots \leq \xi_{l-1} \leq \frac{l-u+1}{v}\right\} = \frac{1}{g!} \left(\frac{s+1}{v}\right)^g,$$

$$V_3 = \text{Vol}\left\{\frac{l-u}{v} \leq \xi_l \leq \frac{l-u+1}{v}\right\} = \frac{1}{v},$$

$$\begin{aligned} V_4 &= \text{Vol}\{\xi_{l+1} \leq \cdots \leq \xi_k \leq 1 : \xi_i \geq \frac{i-u}{v} \forall i\} \\ &= \frac{1}{(k-l)!} \left(\frac{u+v-l}{v}\right)^{k-l} Q_{k-l}(0, u+v-l) \\ &\ll \left(\frac{u+v-l}{v}\right)^{k-l} \frac{(u+v-k)^2}{(k-l+1)!}. \end{aligned}$$

Thus

$$\text{Vol}(R) \ll \frac{(s+1)^g (u+1) g^2 (u+v-k)^2}{g! v^k (k+1-g)!} \sum_l \binom{k+1-g}{l-g} (l-u+1)^{l-g-1} (u+v-l)^{k-l}.$$

By Lemma 4.2 (with  $t = k+1-g$ ,  $a = g+1-u$ ,  $b = u+v-k-1$ ), the sum on  $l$  is

$$\leq e^4 (v+1)^{k-g} \ll v^{k-g} = \frac{v^k}{k^g} \left(\frac{k}{v}\right)^g \leq v^k 10^g \frac{(k-g+1)!}{k \cdot k!}$$



and the lemma follows.  $\square$

To bound  $U_k(v)$ , we will bound the volume of the set

$$\mathcal{T}(k, v, \gamma) = \{\boldsymbol{\xi} \in \mathbb{R}^k : 0 \leq \xi_1 \leq \dots \leq \xi_k \leq 1, 2^{v\xi_1} + \dots + 2^{v\xi_j} \geq 2^{j-\gamma} \ (1 \leq j \leq k)\}.$$

**Lemma 4.4.** *Suppose  $k, v, \gamma$  are integers with  $1 \leq k \leq 10v$  and  $\gamma \geq 0$ . Set  $b = k - v$ . Then*

$$\text{Vol}(\mathcal{T}(k, v, \gamma)) \ll \frac{Y}{2^{2b-\gamma}(k+1)!}, \quad Y = \begin{cases} b & \text{if } b \geq \gamma + 5 \\ (\gamma + 5 - b)^2(\gamma + 1) & \text{if } b \leq \gamma + 4 \end{cases}.$$

*Proof.* Let  $r = \max(5, b - \gamma)$  and  $\boldsymbol{\xi} \in \mathcal{T}(k, v, \gamma)$ . Then either

$$(4.5) \quad \xi_j > \frac{j-\gamma-r}{v} \quad (1 \leq j \leq k)$$

or

$$(4.6) \quad \min_{1 \leq j \leq k} (\xi_j - \frac{j-\gamma}{v}) = \xi_l - \frac{l-\gamma}{v} \in [\frac{-h}{v}, \frac{1-h}{v}] \text{ for some integers } h \geq r + 1, 1 \leq l \leq k.$$

Let  $V_1$  be the volume of  $\boldsymbol{\xi} \in \mathcal{T}(k, v, \gamma)$  satisfying (4.5). If  $b \geq \gamma + 5$ , (4.5) is not possible, so  $b \leq \gamma + 4$  and  $r = 5$ . By Theorem 4.1,

$$V_1 \leq \frac{Q_k(\gamma + 5, v)}{k!} \ll \frac{(\gamma + 6)(\gamma + 6 - b)^2}{(k + 1)!} \ll \frac{Y}{2^{2b-\gamma}(k + 1)!}.$$

If (4.6) holds, then there is an integer  $m$  satisfying

$$(4.7) \quad m \geq h - 3, \quad 2^m < \frac{l}{2}, \quad \xi_{l-2m} \geq \frac{l-\gamma-2m}{v}.$$

To see (4.7), suppose such an  $m$  does not exist. Then

$$\begin{aligned} 2^{v\xi_1} + \dots + 2^{v\xi_l} &\leq 2 \sum_{l/2 < j \leq l} 2^{v\xi_j} \\ &< 2 \left( 2^{h-3} 2^{l-\gamma-h+1} + \sum_{m \geq h-3} 2^m 2^{l-\gamma-2m} \right) \\ &\leq 2^{l-\gamma}, \end{aligned}$$

a contradiction. Let  $V_2$  be the volume of  $\boldsymbol{\xi} \in \mathcal{T}(k, v, \gamma)$  satisfying (4.6). Fix  $h$  and  $m$  satisfying (4.7) and use Lemma 4.3 with  $u = \gamma + h$ ,  $g = 2^m$ ,  $s = 2m$ . The volume of such  $\boldsymbol{\xi}$  is

$$\begin{aligned} &\ll \frac{(\gamma + h + 1)(\gamma + h - b)^2}{(k + 1)!} \frac{(20m + 10)^{2m}}{(2^m - 2)!} \\ &\ll \frac{(\gamma + h + 1)(\gamma + h - b)^2}{2^{2m+3}(k + 1)!}. \end{aligned}$$

The sum of  $2^{-2m+3}$  over  $m \geq h - 3$  is  $\ll 2^{-2h}$ . Summing over  $h \geq r + 1$  gives

$$V_2 \ll \frac{(\gamma + r + 2)(\gamma - b + r + 2)^2}{2^{2r+1}(k + 1)!} \ll \frac{Y}{2^{2b-\gamma}(k + 1)!}.$$

$\square$

*Proof of Lemma 3.4.* Assume  $k \geq 1$ , since the lemma is trivial when  $k = 0$ . Put  $b = k - v$  and define

$$F(\boldsymbol{\xi}) = \min_{0 \leq j \leq k} 2^{-j} (2^{v\xi_1} + \dots + 2^{v\xi_j} + 1).$$

For integers  $m \geq 0$ , consider  $\boldsymbol{\xi} \in R_k$  satisfying  $2^{-m} \leq F(\boldsymbol{\xi}) < 2^{1-m}$ . For  $1 \leq j \leq k$  we have

$$2^{-j} (2^{v\xi_1} + \dots + 2^{v\xi_j}) \geq \max(2^{-j}, 2^{-m} - 2^{-j}) \geq 2^{-m-1},$$

so  $\boldsymbol{\xi} \in \mathcal{T}(k, v, m+1)$ . Hence, by Lemma 4.4,

$$U_k(v, g) \leq \sum_{m \geq 0} 2^{1-m} \text{Vol}(\mathcal{T}(k, v, m+1)) \ll \frac{1}{(k+1)!} \sum_{m \geq 0} \frac{2^{-m} Y_m}{2^{2^{b+t-m-1}}},$$

$$Y_m = \begin{cases} b & \text{if } m \leq b-6 \\ (m+6-b)^2(m+2) & \text{if } m \geq b-5 \end{cases}.$$

Next,

$$\sum_{m \geq 0} \frac{2^{-m} Y_m}{2^{2^{b-m-1}}} = \sum_{0 \leq m \leq b-6} \frac{b}{2^m 2^{2^{b-m-1}}} + \sum_{m \geq \max(0, b-5)} \frac{(m+6-b)^2(m+2)}{2^m}.$$

The proof is completed by noting that if  $b \geq 6$ , each sum on the right side is  $\ll b2^{-b}$  and if  $b \leq 5$ , the first sum is empty and the second is  $\ll (6-b)^2 \ll 1+b^2$ .  $\square$

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