THE BRUN-HOOLEY SIEVE

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1. Introduction

The object of this note is to give an alternative and, we think, simpler account of the Brun-Hooley sieve (see [Ho]) and to derive a general theorem that is in a form ready for numerous applications. We shall put forward also a 'dual' form of Hooley's method that probably has relevance to the multi-dimensional vector sieve of Brüdern and Fouvry ([BF1],[BF2]).

Let $\mathcal{A}$ denote a finite integer sequence of about X elements and let $\mathcal{P}$ be a finite set of primes. Writing $P = \prod_{p\in\mathcal{P}} p$ and (a,b) for the highest common factor of a and b, our objective is to estimate the counting number

$$S(\mathcal{A},\mathcal{P}) := |\{a \in \mathcal{A} : (a,P) = 1\}|.$$

The indicator function of the sub-set of $\mathcal{A}$ whose cardinality is $S(\mathcal{A},\mathcal{P})$ is

$$\sum_{d|(a,P)} \mu(d), \quad a \in \mathcal{A};$$

and it is well known from Brun's 'pure' sieve (a special case of the inclusion-exclusion inequalities) that if $\nu(d)$ denotes the number of prime divisors of d and k is an even natural number, then

$$(1) \qquad \sum_{d|(a,P)} \mu(d) \le \sum_{\substack{d|(a,P)\\ \nu(d)\le k}} \mu(d).$$

Now let

$$\mathcal{P} = \bigcup_{j=1}^{r} \mathcal{P}_j$$

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be a partition of $\mathcal{P}$ (so that $\mathcal{P}_i \cap \mathcal{P}_j = \phi$ if $i \neq j$) and write $P_j = \prod_{p\in\mathcal{P}_j} p$. Then, following Hooley (equivalently Bonferroni's inequalities),

$$\sum_{d|(a,P)} \mu(d) = \prod_{j=1}^{r} \sum_{d|(a,P_j)} \mu(d)$$

$$\leq \prod_{j=1}^{r} \sum_{\substack{d|(a,P_j)\\ \nu(d)\leq k_j}} \mu(d). \tag{2}$$

for any choice of r positive even integers $k_1, \ldots, k_r$; and consequently

$$\begin{aligned} S(\mathcal{A},\mathcal{P}) &= \sum_{a\in\mathcal{A}} \sum_{d|(a,P)} \mu(d) \\ &\leq \sum_{\substack{d_1,\ldots,d_r\\ d_j|P_j,\nu(d_j)\leq k_j}} \mu(d_1)\cdots\mu(d_r)|\{a\in\mathcal{A} : d_1\ldots d_r \mid a\}|. \end{aligned} \tag{3}$$

In Brun's 'pure' sieve the inequality in (1) is reversed if k is odd, but for $r \geq 2$ there is no such simple counterpart to (2). To find a lower bound for $S(\mathcal{A},\mathcal{P})$ Hooley derives an identity that is rather complicated to prove and to state, but we can reach much the same conclusion via the following simple inequality:

Lemma 1. *Suppose that* $0 \leq x_j \leq y_j$ $(j = 1, \ldots, r)$. *Then*

$$x_1 \ldots x_r \geq y_1 \ldots y_r - \sum_{\ell=1}^{r} (y_\ell - x_\ell) \prod_{\substack{j=1\\ j\neq\ell}}^{r} y_j. \tag{4}$$

Proof. The inequality holds (with equality) when $r = 1$, and follows by induction on r from

$$\begin{aligned} y_1 \ldots y_r - x_1 \ldots x_r &= (y_1 \ldots y_{r-1} - x_1 \ldots x_{r-1})y_r + x_1 \ldots x_{r-1}(y_r - x_r) \\ &\leq (y_1 \ldots y_{r-1} - x_1 \ldots x_{r-1})y_r + y_1 \ldots y_{r-1}(y_r - x_r). \quad \square \end{aligned}$$

We apply the inequality with

$$x_j = \sum_{d|(a,P_j)} \mu(d), \quad y_j = \sum_{\substack{d|(a,P_j)\\ \nu(d)\leq k_j}} \mu(d) \quad (j = 1, \ldots, r);$$

from Brun's 'pure' sieve (see for example, [HR], Chapter 2, (2.4))

$$0 \leq y_\ell - x_\ell \leq \sum_{\substack{d|(a,P_\ell)\\ \nu(d)=k_\ell+1}} 1 \quad (\ell = 1, \ldots, r), \tag{5}$$

whence, by (4),

$$\sum_{d|(a,P)} \mu(d) \geq \prod_{j=1}^{r} \Bigg(\sum_{\substack{d|(a,P_j)\\ \nu(d)\leq k_j}} \mu(d) \Bigg) - \sum_{\ell=1}^{r} \Bigg(\sum_{\substack{d|(a,P_\ell)\\ \nu(d)=k_\ell+1}} 1 \Bigg) \prod_{\substack{j=1\\ j\neq \ell}}^{r} \Bigg(\sum_{\substack{d|(a,P_j)\\ \nu(d)\leq k_j}} \mu(d) \Bigg)$$

and therefore (cf. (3))

$$\begin{aligned} S(\mathcal{A},\mathcal{P}) \geq & \sum_{\substack{d_1,\dots,d_r\\ d_j|P_j,\nu(d_j)\leq k_j}} \mu(d_1)\cdots\mu(d_r)\,|\{a\in\mathcal{A} : d_1\cdots d_r \mid a\}| \\ & - \sum_{\ell=1}^{r} \sum_{\substack{d_1,\dots,d_r\\ d_j|P_j,\nu(d_j)\leq k_j\,(j\neq\ell)\\ d_\ell|P_\ell,\nu(d_\ell)=k_\ell+1}} \mu\Bigg(\frac{d_1\cdots d_r}{d_\ell}\Bigg)\,|\{a\in\mathcal{A} : d_1\cdots d_r \mid a\}|\,. \end{aligned} \tag{6}$$

The proof of (5) is quite simple but, in any case, (5) will appear as a very special case of a certain general identity ([DHR], Lemma 2.1) which we shall prove next.

2. A sieve identity

For each integer d let $p^-(d)$, $p^+(d)$ denote the least and largest prime factors respectively of d, and set $p^+(1) = 1$. Next, let $\chi(d)$ be any function defined on the set of all positive integer divisors d of P that has the following properties: (i) $\chi(1) = 1$; (ii) $\chi(d)$ assumes only the values 0 or 1; (iii) χ is divisor-closed in the sense that if $\chi(d) = 1$ and $t \mid d$, then $\chi(t) = 1$. Associate with χ its 'complementary' function $\bar\chi(\cdot)$ given by

$$\bar\chi(1) = 0, \quad \bar\chi(d) = \chi(d/p^-(d)) - \chi(d) \quad \text{when } d > 1,\ d \mid P.$$

Note that $\bar\chi(d)$ also assumes only the values 0 or 1 and that $\bar\chi(d) = 0$ when $\chi(d) = 1$.

Example. *Let*

$$\chi(d) = \chi^{(k)}(d) = \begin{cases} 1, & \nu(d) \leq k, \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\bar\chi^{(k)}(d) = 1 \quad \textit{if and only if } \nu(d) = k+1.$$

The identity we mentioned earlier first occurs in [HR], Chapter 2, §1, and is sometimes referred to as the "fundamental sieve identity"; it asserts that

Lemma 2. *For any divisor D of P and any arithmetic function $h(\cdot)$,*

$$\sum_{d|D} h(d) = \sum_{d|D} h(d)\chi(d) + \sum_{d|D} \bar{\chi}(d) \sum_{\substack{t|D \\ p^+(t)<p^-(d)}} h(dt) \tag{7}$$

(note that, in the second sum on the right, $d > 1$ may be assumed since $\bar{\chi}(1) = 0$). In particular, if h is multiplicative,

$$\sum_{d|D} h(d) = \sum_{d|D} h(d)\chi(d) + \sum_{d|D} h(d)\bar{\chi}(d) \prod_{\substack{p|D \\ p<p^-(d)}} (1 + h(p)). \tag{8}$$

Before we prove the identity we shall illustrate it by taking $h = \mu$. Since

$$\prod_{\substack{p|D \\ p<p^-(d)}} (1 + \mu(p)) = \begin{cases} 1, & p^-(d) = p^-(D), \\ 0 & \text{otherwise}, \end{cases}$$

we obtain

$$\sum_{d|D} \mu(d) = \sum_{d|D} \mu(d)\chi(d) + \sum_{\substack{d|D \\ p^-(d)=p^-(D)}} \mu(d)\bar{\chi}(d), \tag{9}$$

and it follows in particular from the above example that

$$\sum_{d|D} \mu(d) = \sum_{\substack{d|D \\ \nu(d)\le k}} \mu(d) + (-1)^{k+1} \sum_{\substack{d|D \\ p^-(d)=p^-(D) \\ \nu(d)=k+1}} 1,$$

so that (1) and (5) follow.

Proof of the Identity (from [DHR]). Suppose $d > 1$ is any divisor of D, and write

$$d = p_1 \cdots p_m, \quad p_1 > p_2 > \cdots > p_m.$$

Then

$$1 - \chi(d) = \sum_{i=1}^{m} (\chi(p_1 \cdots p_{i-1}) - \chi(p_1 \cdots p_i)) = \sum_{i=1}^{m} \bar{\chi}(p_1 \cdots p_i)$$
$$= \sum_{\substack{\delta|d, \delta>1 \\ p^+(d/\delta)<p^-(\delta)}} \bar{\chi}(\delta),$$

and therefore

$$\sum_{d|D} h(d)(1 - \chi(d)) = \sum_{\substack{d|D \\ d>1}} h(d) \sum_{\substack{\delta|d, \delta>1 \\ p^+(d/\delta)<p^-(\delta)}} \bar{\chi}(\delta) = \sum_{\delta|D, \delta>1} \bar{\chi}(\delta) \sum_{\substack{\delta t|D \\ p^+(t)<p^-(\delta)}} h(\delta t)$$
$$= \sum_{\delta|D, \delta>1} \bar{\chi}(\delta) \sum_{\substack{t|D \\ p^+(t)<p^-(\delta)}} h(\delta t).$$

This proves (7), and for multiplicative h (8) is obvious. □

3. The main result

To progress beyond (3) and (6) we postulate some information about $|\{a \in \mathcal{A} : d \mid a\}|$ when $d \mid P$; and it is usual to assume that there exists a non-negative multiplicative arithmetic function $\omega(\cdot)$ such that the numbers

$$r_d := |\{a \in \mathcal{A} : d \mid a\}| - \frac{\omega(d)}{d} X$$

are in some sense remainders (note that $r_1 = |\mathcal{A}| - X$). Then, by (3),

$$S(\mathcal{A}, \mathcal{P}) \leq \Pi X + R \tag{10}$$

where

$$\Pi := \prod_{j=1}^{r} \Bigg(\sum_{\substack{d \mid P_j \\ \nu(d) \leq k_j}} \mu(d) \frac{\omega(d)}{d} \Bigg) \quad \text{and } R := \sum_{\substack{d_1, \dots, d_r \\ d_j \mid P_j, \nu(d_j) \leq k_j}} |r_{d_1 \cdots d_r}|; \tag{11}$$

and similarly (6) leads to

$$S(\mathcal{A}, \mathcal{P}) \geq \Pi \Bigg\{ 1 - \sum_{\ell=1}^{r} \Bigg(\sum_{\substack{d \mid P_\ell \\ \nu(d) = k_\ell + 1}} \frac{\omega(d)}{d} \Bigg) U_\ell^{-1} \Bigg\} X - R - R' \tag{12}$$

where

$$U_\ell := \sum_{\substack{d \mid P_\ell \\ \nu(d) \leq k_\ell}} \mu(d) \frac{\omega(d)}{d} \quad (\ell = 1, \dots, r) \tag{13}$$

and

$$R' := \sum_{\ell=1}^{r} \sum_{\substack{d_1, \dots, d_r \\ \nu(d_j) \leq k_j \ (j \neq \ell) \\ \nu(d_\ell) = k_\ell + 1}} |r_{d_1 \cdots d_r}|. \tag{14}$$

Write

$$W_j = \sum_{d \mid P_j} \mu(d) \frac{\omega(d)}{d} = \prod_{p \in \mathcal{P}_j} \left(1 - \frac{\omega(p)}{p} \right)$$

and

$$W = \sum_{d \mid P} \mu(d) \frac{\omega(d)}{d} = \prod_{p \in \mathcal{P}} \left(1 - \frac{\omega(p)}{p} \right) = W_1 W_2 \cdots W_r.$$

We expect $S(\mathcal{A},\mathcal{P})$ to be comparable (in some sense) with XW. Apply (8) with $D = P_j$, $h(d) = \mu(d)\frac{\omega(d)}{d}$ and $\chi = \chi^{(k_j)}$ to deduce that

$$W_j = \sum_{\substack{d|P_j \\ \nu(d)\le k_j}} \mu(d)\frac{\omega(d)}{d} + (-1)^{k_j+1} \sum_{\substack{d|P_j \\ \nu(d)=k_j+1}} \frac{\omega(d)}{d} \prod_{\substack{p\in\mathcal{P}_j \\ p<p^-(d)}} \left(1-\frac{\omega(p)}{p}\right),$$

whence, for each $j = 1, \ldots, r$, since each k_j is even, we have

$$U_j - \sum_{\substack{d|P_j \\ \nu(d)=k_j+1}} \frac{\omega(d)}{d} \le W_j \le U_j - W_j \sum_{\substack{d|P_j \\ \nu(d)=k_j+1}} \frac{\omega(d)}{d}. \tag{15}$$

Also

$$\sum_{\substack{d|P_j \\ \nu(d)=k_j+1}} \frac{\omega(d)}{d} \le \frac{1}{(k_j+1)!}\left(\sum_{p\in\mathcal{P}_j} \frac{\omega(p)}{p}\right)^{k_j+1}, \tag{16}$$

and

$$\sum_{p\in\mathcal{P}_j} \frac{\omega(p)}{p} \le \sum_{p\in\mathcal{P}_j} \log\left(1-\frac{\omega(p)}{p}\right)^{-1} = \log W_j^{-1} =: L_j, \tag{17}$$

say. Hence, by (11), (15) and (16),

$$W \le \Pi \le W \prod_{j=1}^{r} \left(1 + \frac{L_j^{k_j+1}}{(k_j+1)!}e^{L_j}\right) \le W \exp E \tag{18}$$

on writing

$$E := \sum_{j=1}^{r} \frac{L_j^{k_j+1}}{(k_j+1)!}e^{L_j}; \tag{19}$$

and by (11) it follows that

$$S(\mathcal{A},\mathcal{P}) \le XW \exp E + R. \tag{20}$$

Next we turn to (12). By (15),

$$U_\ell^{-1} \le W_\ell^{-1}(1+V_\ell)^{-1}, \quad V_\ell := \sum_{\substack{d|P_\ell \\ \nu(d)=k_\ell+1}} \frac{\omega(d)}{d},$$

so that, using (17) and (18),

$$S(\mathcal{A},\mathcal{P}) \geq \{1 - E'\}X\Pi - R - R' \geq \{1 - E'\}XW - R - R', \tag{21}$$

where

$$E' := \sum_{j=1}^{r} \frac{e^{L_j}}{1 + L_j^{-1-k_j}(k_j+1)!}. \tag{22}$$

Since $E' < E$ we obtain the less precise but simpler bound

$$S(\mathcal{A},\mathcal{P}) \geq \{1 - E\}XW - R - R'. \tag{23}$$

To sum up:

Theorem. *With E, E', R and R' as defined in (19), (22), (11) and (14), respectively, we have*

$$S(\mathcal{A},\mathcal{P}) \leq X \prod_{p\in\mathcal{P}} \left(1 - \frac{\omega(p)}{p}\right) \exp E + R$$

and

$$S(\mathcal{A},\mathcal{P}) \geq (1 - E')X \prod_{p\in\mathcal{P}} \left(1 - \frac{\omega(p)}{p}\right) - R - R'$$
$$\geq (1 - E)X \prod_{p\in\mathcal{P}} \left(1 - \frac{\omega(p)}{p}\right) - R - R'.$$

From now on take $\mathcal{P}$ to be a set of primes in the interval $[2, z)$ and for each $j = 1, 2, \ldots, r$ let $\mathcal{P}_j = \mathcal{P} \cap [z_{j+1}, z_j)$ where

$$2 = z_{r+1} < z_r < \cdots < z_1 = z.$$

For the moment we also assume, as is often the case, that

$$|r_d| \leq \omega(d), \quad d|P. \tag{23}$$

Then

$$\sum_{\substack{d|P_j\\ \nu(d)\leq k_j}} \omega(d) < z_j^{k_j} \sum_{d|P_j} \omega(d)/d = z_j^{k_j} \prod_{p\in\mathcal{P}_j} \left(1 + \frac{\omega(p)}{p}\right) \leq z_j^{k_j} W_j^{-1}$$

and hence, by (11),

$$R < \Big(\prod_{j=1}^{r} z_j^{k_j}\Big) W^{-1}.$$

Similarly,

$$R' < \Big(\prod_{j=1}^{r} z_j^{k_j}\Big) W^{-1} \sum_{\ell=1}^{r} z_\ell W_\ell V_\ell < z\Big(\prod_{j=1}^{r} z_j^{k_j}\Big) W^{-1} \sum_{\ell=1}^{r} \frac{L_\ell^{k_\ell+1}}{(k_\ell+1)!}$$
$$< z\Big(\prod_{j=1}^{r} z_j^{k_j}\Big) W^{-1} E$$

by (16), (17) and (19). We conclude that

Corollary. *With a partition of $\mathcal{P}$ of the kind described above, and assuming only the condition (23), we have*

$$S(\mathcal{A},\mathcal{P}) \le XW\{\exp E + \eta\}, \tag{24}$$

where

$$\eta = \Big(\prod_{j=1}^{r} z_j^{k_j}\Big) X^{-1} W^{-2};$$

and that

$$S(\mathcal{A},\mathcal{P}) \ge XW\Big\{1 - E' - \eta - \eta z E\Big\}. \tag{25}$$

We also consider another type of bound on the remainders r_d, by supposing that $|\mathcal{A}| = \pi(Y)$, the number of primes $\le Y$, and for each $d|P$, there are $s(d)$ numbers $t_1, \ldots, t_{s(d)}$ so that

$$|\{a \in \mathcal{A} : d|a\}| = \sum_{h=1}^{s(d)} \pi(Y; d, t_h),$$

where $\pi(Y; d, t)$ is the number of primes $\le Y$ in the residue class t mod d. Here $\omega(d) = ds(d)/\phi(d)$ (in particular $s(d)$ must be multiplicative) and

$$|r_d| \le \sum_{h=1}^{s(d)} \left| \pi(Y; d, t_h) - \frac{\pi(Y)}{\phi(d)} \right|.$$

The quantities R and R' are then bounded using the Bombieri-A.I. Vinogradov Theorem. For every $B > 0$ there is a number A so that the following holds. If

$$\prod_{j=1}^{r} z_j^{k_j} \leq Y^{1/2}(\log Y)^{-A},$$

then $R \ll Y(\log Y)^{-B}$ and thus

$$S(\mathcal{A}, \mathcal{P}) \leq XW \exp E + O(Y(\log Y)^{-B}), \tag{26}$$

and if

$$z \prod_{j=1}^{r} z_j^{k_j} \leq Y^{1/2}(\log Y)^{-A},$$

then $R + R' \ll Y(\log Y)^{-B}$ and

$$S(\mathcal{A}, \mathcal{P}) \geq XW(1 - E') - O(Y(\log Y)^{-B}). \tag{27}$$

For an appropriate choice of B, R and R' will be of smaller order than XW.

Remark. Michael Filaseta has pointed out to us that the Brun-Hooley sieve in the above form may also be applied to a more general type of sieve. If $\mathcal{A}$ is any finite set we may associate with each prime $p \in \mathcal{P}$ a subset $\mathcal{A}_p$. All of the above inequalities hold if we replace the quantity (a, P) by

$$\prod_{a \in \mathcal{A}_p} p$$

throughout.

4. Applications

Inequalities (24)–(27) yield three kinds of results. We will concentrate on (24) and (25) for now, as the same type of bounds also follow from (26) and (27) in a similar fashion.

I. By (24),

$$S(\mathcal{A}, \mathcal{P}) \ll XW$$

provided only that E and η are bounded. This estimate has numerous applications as an auxiliary counting device.

II. Inequality (25) is non-trivial only if

$$E' + \eta + \eta z E < 1,$$

for example, if $E' < 1$ and $\eta z E = o(1)$ as $X \to \infty$. Then

$$S(\mathcal{A}, \mathcal{P}) > 0$$

tells us that there exists an element a of $\mathcal{A}$ all of whose prime factors from $\mathcal{P}$ are large; and if $\mathcal{P}$ is carefully chosen it will follow that a has very few prime factors in all. We shall give illustrations below.

III. Together, (24) and (25) yield

$$S(\mathcal{A},\mathcal{P}) \sim XW \quad \text{as} \quad X \to \infty,$$

provided that $z\eta$ is bounded and $E = o(1)$ as $X \to \infty$. This is a result of 'fundamental lemma' type, and also has numerous applications.

We make all this clearer by choosing the sub-division points z_j and postulating some further information about the function ω. Let

$$z_r = \log\log X =: \xi \tag{28}$$

for short and

$$\log z_j = K^{1-j} \log z \quad (j = 1, \ldots, r-1) \tag{29}$$

where $K > 1$ is a constant to be chosen conveniently. Of course we regard X as very large, and we determine r uniquely by

$$z^{K^{1-r}} \le \xi < z_{r-1} = z^{K^{2-r}},$$

so that, in particular,

$$\frac{1}{\log \xi} \le \frac{K^{r-1}}{\log z} < \frac{K}{\log \xi}. \tag{30}$$

We defer the choice of the even integers k_j except that we put $k_r = \infty$ always. This is in order provided we estimate the magnitude of a divisor d of P_r by $d < \xi^{\pi(\xi)} < \xi^{\xi}$ in place of ξ^{k_r}. As a consequence we have to modify the definition of η to

$$\eta = \left(\prod_{j=1}^{r-1} z_j^{k_j} \right) \xi^{\xi} X^{-1} W^{-2}, \tag{31}$$

and also note that, in the definitions (19) and (22) of E and E', the summation over j now runs from 1 to $r-1$ only.

Next we impose on $\omega(\cdot)$ the well-known Iwaniec condition:

(Ω) *Suppose there exist positive constants κ and A such that*

$$\prod_{y_1 \le p < y_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \le \left(\frac{\log y_2}{\log y_1}\right)^{\kappa} \exp\left(\frac{A}{\log y_1}\right), \quad 2 \le y_1 < y_2.$$

Then

$$W^{-1} \leq \left(\frac{\log z}{\log 2}\right)^{\kappa} \exp\left(\frac{A}{\log 2}\right) = B(\log z)^{\kappa}, \quad B = \frac{\exp(A/\log 2)}{(\log 2)^{\kappa}}, \tag{32}$$

and, by (17),

$$L_j \leq \kappa \log\left(\frac{\log z_j}{\log z_{j+1}}\right) + \frac{A}{\log z_{j+1}} = \kappa \log K + \frac{AK^j}{\log z} \quad (1 \leq j \leq r-1),$$

so that, by (30),

$$L_j < \kappa \log K + \frac{AK}{\log \xi} =: L \quad (1 \leq j \leq r-1), \tag{33}$$

say.

Let us write

$$z = X^{1/u}, \quad u > 1;$$

then, by (31),

$$\eta \leq B^2 X^{\frac{\Gamma}{u}-1} (\log X)^{2\kappa/u + \log \xi}, \quad \Gamma := \sum_{j=1}^{r-1} \frac{k_j}{K^{j-1}}. \tag{34}$$

Also, by (19)

$$E < e^L \sum_{j=1}^{r-1} \frac{L^{k_j+1}}{(k_j+1)!} \tag{35}$$

and by (22)

$$E' < e^L \sum_{j=1}^{r-1} \frac{1}{1 + L^{-k_j-1}(k_j+1)!}. \tag{36}$$

We see from (34) that

$$z\eta = o(1) \quad \text{as } X \to \infty \text{ if } \Gamma + 1 < u. \tag{37}$$

Choosing the even integers $k_1 \ldots, k_{r-1}$ depends on the kind of application one has in mind. In categories I and III a reasonable all-purpose choice is

$$k_j = b + 2(j-1), \quad j = 1, \ldots, r-1,$$

where $b \geq 2$ is an even integer that remains at our disposal. Here

$$\Gamma = \sum_{i=0}^{r-2} \frac{b+2i}{K^i} < \frac{bK}{K-1} + \frac{2K}{(K-1)^2} \,,$$

so that $z\eta = o(1)$ if

$$u > 1 + \frac{bK^2 - (b-2)K}{(K-1)^2} \,;$$

also, by (35) (and bearing in mind an earlier remark)

$$\begin{aligned} E &\leq e^L \sum_{j=1}^{r-1} \frac{L^{b+1+2(j-1)}}{(b+1+2(j-1))!} = e^L \sum_{i=0}^{r-2} \frac{L^{b+1+2i}}{(b+1+2i)!} \\ &\leq \frac{L^{b+1}}{(b+1)!} e^L \sum_{i=0}^{\infty} \frac{L^{2i}}{(2i)!} < \frac{L^{b+1}}{(b+1)!} e^{2L} < \left(\frac{eL}{b+1}\right)^{b+1} e^{2L}. \end{aligned}$$

By (33), $L < 1.01\kappa \log K$ if x is large enough. Taking $K = e^{150/101}$ and $b = 2$ we see that $z\eta = o(1)$ as $X \to \infty$ if $u > 4.35$, and that

$$E < \left(\frac{1}{2}\kappa e^{1+\kappa}\right)^3.$$

This suffices for applications of type I.

For applications of type III we choose b large. For example, take $K = 2 + \sqrt{2}$ and

$$b = 2([\xi] + 1) > 2\xi,$$

so that $z\eta = o(1)$ if $u > 5\xi$ and

$$E < \left(\frac{1.69\kappa}{\xi}\right)^{2\xi} e^{2.49\kappa} \to 0 \quad \text{as } X \to \infty.$$

Notice that here we sieve only up to $z = X^{\frac{1}{4 \log\log X}}$, but obtain asymptotic equality for $S(\mathcal{A}, \mathcal{P})$.

For applications of type II we have to proceed more carefully in order to arrive at the best results of which the method is capable. Specifically, we have to choose $k_1, \ldots, k_{r-1}$ and K so as to *minimize*

(38) $$1 + \Gamma = 1 + \sum_{j=1}^{r-1} \frac{k_j}{K^{j-1}}$$

subject to

$$e^L \sum_{j=1}^{r-1} \frac{1}{1+(k_j+1)!L^{-1-k_j}} < 1. \tag{39}$$

The best procedure in this optimization exercise is, given a candidate K, to take as many k_j as possible to be 2 (as many as (39) allows), then take as many as possible to be 4, etc. By (33), it is in order to take $L = \kappa \log K$ for purposes of numerical computation, so that $e^L = K^\kappa$. With a candidate K and

$$b(k) := \frac{K^\kappa}{1+(k+1)!(\kappa \log K)^{-k-1}},$$

the explicit procedure is to take the first $n_2 = \lfloor 1/b(2) \rfloor$ k_j's to be 2, the next $n_4 = \lfloor (1 - n_2 b(2))/b(4) \rfloor$ k_j's to be 4, etc. In this way (35) remains true automatically while the candidate K in conjunction with n_2 twos, n_4 fours, etc. determines $1+\Gamma$.

The following example will serve as an illustration.

Example. Let $\mathcal{A} = \{n^2+1 : n \le x\}$ and $\mathcal{P} = \{2\} \cup \{p < z : p \equiv 1 \bmod 4\}$. Here $X = x$, $\omega(2) = 1$, $\omega(p) = 2$ when $p \equiv 1 \bmod 4$ ($\omega(p) = 0$ otherwise), and

$$\prod_{y_1 \le p < y_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} = \prod_{\substack{y_1 \le p < y_2 \\ p \equiv 1 \ (\mathrm{mod}\ 4)}} \left(1 - \frac{2}{p}\right)^{-1}, \quad 2 < y_1 < y_2,$$
$$= \frac{\log y_2}{\log y_1}\left(1 + O\left(\frac{1}{\log y_1}\right)\right).$$

Thus the Iwaniec condition (Ω) holds with $\kappa = 1$.

The best choice of K turns out to be 2.572, and one finds that $n_2 = 3$, $n_4 = 3$, $n_6 = 3$, $n_8 = 67$, etc., and therefore $1 + \Gamma < 4.4766$. Take u to be 4.48 and $z = x^{1/u} = x^{1/4.48}$. We may conclude that $\mathcal{A}$ contains $\gg x/\log x$ elements having no prime factor $< x^{\frac{1}{4.48}}$, and each of these elements obviously cannot have more than 8 prime factors, or, as we say, is a P_8.

The following table summarizes the best choices for $\kappa = 1, 2, 3, 4, 5$.

κ	K	u	k_1
1	2.57200	4.4766	2
2	1.54062	7.7441	2
3	1.28121	11.7710	2
4	1.41012	15.6685	4
5	1.31470	19.3749	4

The interested reader should be able to verify easily, using $\kappa = 2$, that the number of prime twins not exceeding x is $\ll x(\log x)^{-2}$, and that there exist infinitely many integers such that each of n, $n+2$ is the product of at most 7 prime factors. The much more complicated Brun's sieve gives nothing better.

Although dealing with a set $\mathcal{A}$ which is of the form $\{f(p) : p \le X, p \text{ prime}\}$, where f is a polynomial, requires an additional result (the Bombieri-A. I. Vinogradov Theorem), it is still straightforward to obtain bounds in this case. For Type II results, we note that (27) holds provided that $u > 2(\Gamma+1)$, where Γ is given by (38) and we require (39) to hold. For example, if $\mathcal{A} = \{p+2 : p \le X, p \text{ prime}\}$, so that $\kappa = 1$, it follows that for infinitely many primes p, $p+2$ is composed of prime factors $\le X^{1/8.96}$, which implies that $p+2 = P_8$.

We are indebted to the referee for several helpful remarks, and especially for pointing out that the remainder sums R and R' have, potentially, a highly flexible structure – for example, we could leave R in the form

$$\sum_{\substack{d_1,\dots,d_r \\ d_j | P_j, \nu(d_j) \le k_j}} \mu(d_1)\cdots\mu(d_r) r_{d_1\cdots d_r}$$

– and that there are perhaps applications where this would be an advantage, for instance if one were then able to use more recent and sharper versions of the Bombieri-Vinogradov theorem. In the case of the prime twin conjecture, however, any such refinement if deployed above would not improve on what can be accomplished by the more sophisticated Rosser-Iwaniec sieve methods.

5. A dual of Hooley's method

This method in the form of inequality (4) lends itself to a dual purpose. Rather than aim for full generality here, consider the case of

$$\mathcal{A} = \left\{ \prod_{j=1}^{r} (a_j n + b_j) : n \le x \right\}, \qquad r \ge 2,$$

where the a_j, b_j are integers satisfying

$$\prod_{j=1}^{r} a_j \prod_{1 \le i < j \le r} (a_i b_j - a_j b_i) \ne 0,$$

and the polynomial

$$F(n) := \prod_{j=1}^{r} (a_j n + b_j)$$

has no fixed prime divisors. Let $\mathcal{P}$ be the set of all primes truncated at some z. Obviously we are here addressing a generalized prime k-tuples conjecture, and the problem of estimating $S(\mathcal{A},\mathcal{P})$ is of 'dimension' r, that is, has $\kappa = r$. However, following the 'vector' sieve of Brüdern & Fouvry mentioned at the start, we have

$$\begin{aligned}\sum_{d|(F(n),P)} \mu(d) &= \prod_{j=1}^{r} \sum_{d|(a_j n+b_j,P)} \mu(d)\\ &\le \prod_{j=1}^{r} \sum_{d|(a_j n+b_j,P)} \mu(d)\chi^+(d)\end{aligned}$$

where $\chi^+(d)$ characterizes the LINEAR upper Rosser-Iwaniec sieve; and, as in (4),

$$\begin{aligned}\sum_{d|(F(n),P)} \mu(d) \ge& \prod_{j=1}^{r} \sum_{d|(a_j n+b_j,P)} \mu(d)\chi^+(d)\\ &- \sum_{\ell=1}^{r}\Bigg(\sum_{\substack{d|(a_\ell n+b_\ell,P)\\ p^-(d)=p^-((a_\ell n+b_\ell,P))}} \bar{\chi}^+(d)\Bigg)\prod_{\substack{j=1\\ j\ne\ell}}^{r}\Bigg(\sum_{d|(a_j n+b_j,P)} \mu(d)\chi^+(d)\Bigg).\end{aligned}$$

This seems to us superior to Lemma 13 of [BF1] or (2.6) of [BF2] in the treatment of the '$y_\ell - x_\ell$' terms, and should lead to better results.

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