ROUGH INTEGERS WITH A DIVISOR IN A GIVEN INTERVAL

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Abstract. We determine, up to multiplicative constants, the number of integers $n \leqslant x$ that have no prime factor $\leqslant w$ and a divisor in $(y, 2y]$. Our estimate is uniform in x, y, w. We apply this to determine the order of the number of distinct integers in the $N \times N$ multiplication table which are free of prime factors $\leqslant w$, and the number of distinct fractions of the form $\frac{a_1 a_2}{b_1 b_2}$ with $1 \leqslant a_1 \leqslant b_1 \leqslant N$ and $1 \leqslant a_2 \leqslant b_2 \leqslant N$.

1. Introduction

In the paper [**5**], the author established the order of growth of $H(x, y, z)$, the number of integers $n \leqslant x$ which have a divisor in the interval $(y, z]$, for all x, y, z. An important special case is

$$H(x,y,2y) \asymp \frac{x}{(\log y)^{\mathcal{E}}(\log_2 y)^{3/2}} \qquad (3 \leqslant y \leqslant \sqrt{x}), \tag{1.1}$$

where

$$\mathcal{E} = 1 - \frac{1+\log_2 2}{\log 2} = 0.086071332\ldots.$$

A shorter, more direct proof of the order of magnitude bounds in the special case (1.1) is given in [**6**]. More on the history of estimations of $H(x, y, z)$, further applications and references may be found in [**5**].

A number of recent aplications have required similar bounds, but where the underlying set of integers n is restricted to a special set, e.g. the set of shifted primes ([**5**, Theorem 6,7], [**9**]) or the values of a polynomial [**2, 1, 11, 12, 7**]. More generally, we define

$$H(x,y,z;\mathcal{A}) = |\{n \leqslant x, n \in \mathcal{A} : d|n \text{ for some } d \in (y,z]\}|.$$

Another natural set to consider is $\mathcal{R}_w$, the set of integers with no prime factor $p \leqslant w$; called *$w-$rough numbers* by some authors. Here we determinte the exact order of growth of $H(x, y, 2y; \mathcal{R}_w)$ for all x, y, w; the more general quantity $H(x, y, z; \mathcal{R}_w)$ can be estimated by similar methods, although there are many cases depending on the relative size of the parameters w, x, y, z.

Theorem 1. *Suppose that $4 \leqslant y \leqslant \sqrt{x}$, $4 \leqslant w \leqslant y/8$ and write*[1] $\delta = \frac{\log_2 w}{\log_2 y}$.

(i) *When $1 - 1/\log 4 \leqslant \delta \leqslant 1$ we have*

$$H(x,y,2y;\mathcal{R}_w) - H(x/2,y,2y;\mathcal{R}_w) \gg \frac{x}{\log^2 w} \gg H(x,y,2y;\mathcal{R}_w).$$

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[1] The notation $\log_2 x$ stands for $\log\log x$.

(ii) *When* $0 \leqslant \delta < 1 - 1/\log 4$, *we have*

$$H(x,y,2y;\mathcal{R}_w) - H(x/2,y,2y;\mathcal{R}_w) \gg x\delta B(w,y)(\log y)^{-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}} \gg H(x,y,2y;\mathcal{R}_w),$$

where

$$B(w,y) = \min(1, (\log_2 y)^{-1/2}((1-\delta)\log 4 - 1)^{-1}).$$

Remark 1. *Some special cases are worth noting. From Theorem 1 we have*

$$x\delta B(w,y)(\log y)^{-\mathcal{E}+\frac{1-\delta}{\log 2}} \asymp \begin{cases} \frac{x\log_2 w}{(\log_2 y)^{3/2}}(\log y)^{-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}} & (\delta \leqslant 1 - \frac{1}{\log 4} - \varepsilon, \varepsilon > 0 \text{ fixed}) \\ \frac{x\log_2 w}{(\log y)^{\mathcal{E}}(\log w)^{1/\log 2}(\log_2 y)^{3/2}} & (\log_2 w \leqslant \sqrt{\log_2 y}). \end{cases}$$

Remark 2. *When* $y > \sqrt{x}$, *one can obtain similar results by using the duality* $d|n \iff (n/d)|n$. *That is, if* $x/2 < n \leqslant x$, *then* $d|n$ *with* $y < d \leqslant 2y$ *is equivalent to* $d'|n$ *with* $d' \asymp x/y$.

We illustrate the utility of Theorem 1 with two applications. The first is related to the well-know multiplication table problem of Erdős [**3, 4**], which asks for estimates on the number, $M(N)$, of distinct integers in an $N \times N$ multiplication table. In [**5**] the author proved, using (1.1), that

$$M(N) \asymp \frac{N^2}{(\log N)^{\mathcal{E}}(\log_2 N)^{3/2}}. \tag{1.2}$$

More generally, consider the restricted multiplication table problem of bounding $M(N;\mathcal{A})$, the number of distinct entries in an $N \times N$ multiplication table that belong to the set $\mathcal{A}$. For example, when $\lambda \neq 0$ is fixed and $\mathcal{A} = \{p + \lambda : p \text{ prime}\}$, the order of $M(N;\mathcal{A})$ was determined in [**5**, Theorem 6] (upper bound) and [**9**] (lower bound).

Observe that $M(N;\mathcal{R}_w) = 1$ when $w \geqslant N$.

Corollary 2. *Uniformly for* $4 \leqslant w \leqslant N/2$, *we have*

$$M(N;\mathcal{R}_w) \asymp \begin{cases} \frac{N^2}{\log^2 w} & \text{if } \log w \geqslant (\log N)^{1-1/\log 4} \\ N^2\delta B(w,N)(\log N)^{-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}} & \text{if } \log w = (\log y)^{\delta}, \delta \leqslant 1 - \frac{1}{\log 4}. \end{cases}$$

Proof. If $\sqrt{N} < w \leqslant N/2$, then $M(N;\mathcal{R}_w)$ counts entries in the multiplication table which are primes in $(w, N]$ or the product of two such primes. The desired bounds follow. If $4 \leqslant w \leqslant \sqrt{N}$, we use the inequalities

$$H\left(\frac{N^2}{4}, \frac{N}{4}, \frac{N}{2}; \mathcal{R}_w\right) \leqslant M(N;\mathcal{R}_w) \leqslant \sum_{k \geqslant 0} H\left(\frac{N^2}{2^k}, \frac{N}{2^{k+1}}, \frac{N}{2^k}; \mathcal{R}_w\right).$$

The proof is easy: consider $ab \in \mathcal{R}_w, a \leqslant N$ and $b \leqslant N$. If $\frac{N}{4} < a \leqslant \frac{N}{2}$ and $ab \leqslant \frac{N^2}{4}$, then $b \leqslant N$ and this proves the lower bound. The upper bound comes from taking $\frac{N}{2^{k+1}} < a \leqslant \frac{N}{2^k}$ for some non-negative integer k. The desired bound for $M(N;\mathcal{R}_w)$ now follow from Theorem 1, since we have $H(x,y,2y;\mathcal{R}_w) \asymp xf(y,w)$ where $f(u,w) \asymp f(y,w)$ for $\log u \asymp \log y$. □

Next, we consider the "Farey fraction multiplication table". Let $\mathcal{F}_N$ of Farey fractions of order N, i.e.,

$$\mathcal{F}_N = \left\{\frac{a}{b} : 1 \leqslant a \leqslant b \leqslant N, (a,b) = 1\right\}.$$

In private conversation, Igor Shparlinski asked the author about the size of the product set $\mathcal{F}_N\mathcal{F}_N$ (in general, for sets $\mathcal{A},\mathcal{B}\in\mathbb{Z}$, $\mathcal{A}\mathcal{B}$ denotes the product set $\{ab : a\in\mathcal{A}, b\in\mathcal{B}\}$).

Corollary 3. *We have*

$$M(N)^2 \ll |\mathcal{F}_N\mathcal{F}_N| \leqslant M(N)^2.$$

Proof. The upper bound is trivial, and thus the real work is on the lower bound. We achieve this by placing restrictions on the fractions, firstly by putting them in dyadic intervals and secondly by removing those elements divisible by small primes. To this end, define

$$\mathcal{A}_N = \{n : N/2 \leqslant n \leqslant N\}, \quad \mathcal{A}_N^{(w)} = \mathcal{A}_N \cap \mathcal{R}_w. \tag{1.3}$$

Let w be a large, fixed constant. A simple inclusion-exclusion argument yields (here p denotes a prime in the sums)

$$\begin{aligned}|\mathcal{F}_N\mathcal{F}_N| &\geqslant \left|\left\{\frac{a_1a_2}{b_1b_2} : a_1,a_2\in\mathcal{A}_{N/2}^{(w)}; b_1,b_2\in\mathcal{A}_N^{(w)}; (a_1a_2,b_1b_2)=1\right\}\right| \\ &\geqslant |\mathcal{A}_{N/2}^{(w)}\mathcal{A}_{N/2}^{(w)}|\cdot|\mathcal{A}_N^{(w)}\mathcal{A}_N^{(w)}| - \sum_{w<p\leqslant N/2}|\mathcal{A}_{N/2}^{(w)}\mathcal{A}_{N/2p}^{(w)}|\cdot|\mathcal{A}_N^{(w)}\mathcal{A}_{N/p}^{(w)}| \\ &\geqslant |\mathcal{A}_{N/2}^{(w)}\mathcal{A}_{N/2}^{(w)}|\cdot|\mathcal{A}_N^{(w)}\mathcal{A}_N^{(w)}| - \sum_{w<p\leqslant N/2}|\mathcal{A}_{N/2}\mathcal{A}_{N/2p}|\cdot|\mathcal{A}_N\mathcal{A}_{N/p}|.\end{aligned}$$

It is clear that for $M\leqslant N$ we have

$$|\mathcal{A}_N\mathcal{A}_M| \leqslant H(MN, M/2, M)$$

and we deduce from (1.1) that

$$\sum_{w<p\leqslant N/2}|\mathcal{A}_{N/2}\mathcal{A}_{N/2p}|\cdot|\mathcal{A}_N\mathcal{A}_{N/p}| \ll \frac{N^4}{(\log N)^{2\mathcal{E}}(\log_2 N)^3}\sum_{p>w}\frac{1}{p^2} \ll \frac{M(N)^2}{w\log w}.$$

We also have the lower bound

$$|\mathcal{A}_N^{(w)}\mathcal{A}_M^{(w)}| \geqslant H(MN, M/2, M;\mathcal{R}_w) - H(MN/2, M/2, M;\mathcal{R}_w).$$

It follows that

$$\begin{aligned}|\mathcal{F}_N\mathcal{F}_N| \geqslant \Big(H(\tfrac{N^2}{4},\tfrac{N}{4},\tfrac{N}{2};\mathcal{R}_w)-H(\tfrac{N^2}{8},\tfrac{N}{4},\tfrac{N}{2};\mathcal{R}_w)\Big)\Big(H(N^2,\tfrac{N}{2},N;\mathcal{R}_w)-H(\tfrac{N^2}{2},\tfrac{N}{2},N;\mathcal{R}_w)\Big)- \\ - O\left(\frac{N^4}{(\log N)^{2\mathcal{E}}(\log_2 N)^3(w\log w)}\right).\end{aligned} \tag{1.4}$$

Inserting Theorem 1 into the estimate (1.4), and taking w to be a sufficiently large constant, we obtain the lower bound in Corollary 3. □

1.1. **Notation.** Let $\tau(n)$ be the number of positive divisors of n, and $\tau(n;y,z)$ denotes the number of divisors of n within the interval $(y,z]$. Let $\omega(n)$ be the number of distinct prime divisors of n. Let $P^+(n)$ be the largest prime factor of n and let $P^-(n)$ be the smallest prime factor of n. Adopt the notational conventions $P^+(1)=0$ and $P^-(1)=\infty$. Constants implied by O, $\ll$ and $\asymp$ are absolute. The notation $f\asymp g$ means $f\ll g$ and $g\ll f$. The symbol p will always denote a prime. Lastly, $\log_2 x$ denotes $\log\log x$.

1.2. **Heuristics.** Here we give a short heuristic argument to justify the formulas in Theorem 1. This is similar to the heuristics givin in [**5, 6**].

Write $n = n'n''$, where n' is composed only of primes in $(w, 2y]$ and n'' is composed only of primes $> 2y$. For simplicity, assume n' is squarefree and $n' \leqslant y^{100}$. Assume for the moment that the set $D(n') = \{\log d : d|n'\}$ is approximately uniformly distributed in $[0, \log n']$. If n' has k prime factors, then $\tau(n') = 2^k$ and we thus expect that $\tau(n', y, 2y) \geqslant 1$ with probability about

$$\min\left(1, \frac{2^k}{\log y}\right).$$

This expression changes behavior at $k = k_0 := \left\lfloor \frac{\log_2 y}{\log 2} \right\rfloor$. The number of $n \leqslant x$ with $n' \in \mathcal{R}_w$ and $\omega(n') = k$ is of size

$$\frac{x}{\log y} \frac{(\log_2 y - \log_2 w)^k}{k!},$$

and we obtain a heuristic estimate for $H(x, y, 2y; \mathcal{R}_w)$ of order

$$\frac{x}{\log^2 y}\left[\sum_{k\leqslant k_0} \frac{(2\log_2 y - 2\log_2 w)^k}{k!} + (\log y)\sum_{k\geqslant k_0} \frac{(\log_2 y - \log_2 w)^k}{k!}\right].$$

It can be shown that the first sum always dominates, since the second sum is part of the tail of the Poisson distribution (k_0 is always much larger than $\log_2 y - \log_2 w$). The behavior of the first sum over k depends on the relative sizes of k_0 and $2\log_2 y - 2\log_2 w$. If $k_0 > 2\log_2 y - 2\log_2 w$, that is, $\log w \leqslant (\log y)^{1-1/\log 4}$, the first contains the "peak" and we obtain

$$H(x, y, 2y; \mathcal{R}_w) \approx \frac{x}{\log^2 y} e^{2\log_2 y - 2\log_2 w} = \frac{x}{\log^2 w}.$$

For smaller w, we are summing the left tail of the Poisson distribution and standard bounds (see e.g. (2.1)) yield

$$H(x, y, 2y; \mathcal{R}_w) \approx xB(y, w)(\log y)^{-\mathcal{E} + \frac{\log(1-\delta)}{\log 2}}.$$

This latter expression is too large by a factor $1/\delta$, and this stems from the uniformity assumption about $D(n')$, which turns out to be false for all but a proportion δ of these integers. Fluctuations in the distribution of the prime factors of n' lead to clustering of the divisors; more details can be found in [**5, 6**]. As in [**5, 6**], we really should be considering those n' which have nicely distributed divisors, and a useful measure of how nicely distributed the divisors are is the function

$$L(a) = \text{meas}\mathscr{L}(a), \qquad \mathscr{L}(a) = \bigcup_{d|a} [-\log 2 + \log d, \log d).$$

Adjusting our heuristic, we see that the probability that $\tau(n', y, 2y) \geqslant 1$ should be about $L(n')/\log y$, which is $\gg 1/\log y$ on a set of n' of density δ.

2. Preliminaries

Lemma 2.1 ([**6**, Lemma 3.1])**.** *We have*

(i) *$L(a) \leqslant \min(\tau(a)\log 2, \log 2 + \log a)$;*
(ii) *If $(a, b) = 1$, then $L(ab) \leqslant \tau(b)L(a)$;*

(iii) *If* $p_1 < \cdots < p_k$, *then*

$$L(p_1 \cdots p_k) \leqslant \min_{0 \leqslant j \leqslant k} 2^{k-j} (\log(p_1 \cdots p_j) + \log 2).$$

Let $\mathscr{P}(a,b)$ be the set of all squarefree positive integers composed only of primes in $(a,b]$. We adopt the convention that $1 \in \mathscr{P}(a,b)$ for any a,b.

Lemma 2.2. (a) *For* $t \geqslant w \geqslant 2$ *and* $k \geqslant 0$ *we have*

$$\sum_{\substack{a \in \mathscr{P}(w,t) \\ \omega(a)=k}} \frac{1}{a} \leqslant \frac{(\log_2 t - \log_2 w + O(1))^k}{k!}.$$

(b) *For* $t \geqslant w \geqslant 2$ *and* $k \geqslant 1$ *we have*

$$\sum_{\substack{a \in \mathscr{P}(w,t) \\ \omega(a)=k}} \frac{\log a}{a} \ll (1 + \log(t/w)) \frac{(\log_2 t - \log_2 w + O(1))^{k-1}}{(k-1)!}.$$

(c) *For* $2 \leqslant w \leqslant s \leqslant t$, *we have*

$$\sum_{a \in \mathscr{P}(w,t)} \frac{L(a)}{a} \ll \left(\frac{\log t}{\log s}\right)^2 \sum_{a \in \mathscr{P}(w,s)} \frac{L(a)}{a}.$$

Proof. Item (a) is immediate from

$$\sum_{\substack{a \in \mathscr{P}(w,t) \\ \omega(a)=k}} \frac{1}{a} \leqslant \frac{1}{k!} \left(\sum_{w < p \leqslant t} \frac{1}{p} \right)^k$$

and Mertens' estimate. For item (b), we have

$$\sum_{\substack{a \in \mathscr{P}(w,t) \\ \omega(a)=k}} \frac{\log a}{a} = \sum_{\substack{a \in \mathscr{P}(w,t) \\ \omega(a)=k}} \frac{1}{a} \sum_{p \mid a} \log p \leqslant \sum_{w < p \leqslant t} \frac{\log p}{p} \sum_{\substack{a \in \mathscr{P}(w,t) \\ \omega(a)=k-1}} \frac{1}{a}.$$

The desired inequality follows from part (a) and Mertens' estimates. For part (c), we factor each $a \in \mathscr{P}(w,t)$ uniquely as $a = a_1 a_2$ with $a_1 \in \mathscr{P}(w,s)$ and $a_2 \in \mathscr{P}(s,t)$. Then, using Lemma 2.1 (ii) we deduce that

$$\begin{aligned} \sum_{a \in \mathscr{P}(w,t)} \frac{L(a)}{a} &\leqslant \sum_{a_1 \in \mathscr{P}(w,s)} \frac{L(a_1)}{a_1} \sum_{a_2 \in \mathscr{P}(s,t)} \frac{\tau(a_2)}{a_2} \\ &= \prod_{s < p \leqslant t} \left(1 + \frac{2}{p}\right) \sum_{a_1 \in \mathscr{P}(w,s)} \frac{L(a_1)}{a_1}. \end{aligned}$$

The desired inequality follows from Mertens' estimates. □

The following is a standard sieve bound, see e.g. [**8**].

Lemma 2.3. *(a) Uniformly for $x \geqslant 2z \geqslant 4$, we have*

$$|\{x/2 < n \leqslant x : P^-(n) > z\}| \gg \frac{x}{\log z}.$$

Uniformly for $x \geqslant z \geqslant 2$ we have

$$|\{n \leqslant x : P^-(n) > z\}| \ll \frac{x}{\log z}.$$

We also record a consequence of Stirling's formula:

$$\sum_{h \leqslant k \leqslant m} \frac{x^k}{k!} \asymp \min\left(\sqrt{x}, \frac{x}{x-m}, m-h+1\right) \frac{x^m}{m!} \quad (h \leqslant m \leqslant x). \tag{2.1}$$

3. Local-to-global estimates

Following a kind of local-to-global principle first utilized in [**5**], we bound $H(x, y, 2y; \mathcal{R}_w)$ in terms of the function $L(a)$. This justifies the heuristic presented in Section 1.2.

Lemma 3.1. *If $4 \leqslant w \leqslant y^{1/15}$ and $y \leqslant \sqrt{x}$, then*

$$H(x, y, 2y; \mathcal{R}_w) - H(x/2, y, 2y; \mathcal{R}_w) \gg \frac{x}{\log^2 y} \sum_{a \in \mathscr{P}(w,y)} \frac{L(a)}{a}.$$

If $4 \leqslant w \leqslant y \leqslant \sqrt{x}$ and $w_0 \leqslant w \leqslant y^{1/10}$ for a sufficiently large constant w_0, then

$$H(x, y, 2y; \mathcal{R}_w) \ll \frac{x}{\log^2 y} \sum_{a \in \mathscr{P}(w,y)} \frac{L(a)}{a}.$$

Proof. We begin with the lower bound. Consider integers $n = ap_1p_2b \in (x/2, x]$ with $P^-(a) > w$, p_1 and p_2 prime, satisfying the inequalities

$$a \leqslant y^{1/5} < p_1 < p_2 \leqslant \frac{1}{4} y^{4/5} < P^-(b),$$

and with $\log(y/p_1p_2) \in \mathscr{L}(a)$. The last condition implies that $\tau(ap_1p_2, y, 2y) \geqslant 1$, and we also have that $P^-(n) > w$. Since $y^{4/5} \leqslant y/a < p_1p_2 \leqslant 2y$, we have $x/ap_1p_2 \geqslant x/(2y^{6/5}) \geqslant \frac{1}{2} y^{4/5}$. Thus, by Lemma 2.3, for each triple (a, p_1, p_2), the number of possible b is $\gg \frac{x}{ap_1p_2 \log y}$. Now $\mathscr{L}(a)$ is the disjoint union of intervals of length $\geqslant \log 2$ contained in $[-\log 2, \log a]$. For each such interval $[u, v]$, Mertens' estimate implies that

$$\sum_{\substack{u \leqslant \log(y/p_1p_2) \leqslant v \\ y^{1/5} < p_1 < p_2 < \frac{1}{4} y^{4/5}}} \frac{1}{p_1p_2} \geqslant \sum_{8y^{1/5} < p_1 < y^{2/5}} \frac{1}{p_1} \sum_{ye^{-v}/p_1 < p_2 \leqslant ye^{-u}/p_1} \frac{1}{p_2} \gg \frac{v-u}{\log y}.$$

Here we made use of the estimate $v \leqslant \log a \leqslant \frac{1}{5} \log y$ which implies that $ye^{-v}/p_1 \geqslant y^{2/5} > p_1$. Thus, with a fixed, the sum of $\frac{1}{p_1p_2}$ is $\gg \frac{L(a)}{\log y}$ and we obtain

$$H(x, y, 2y; \mathcal{R}_w) - H(x/2, y, 2y; \mathcal{R}_w) \gg \frac{x}{\log^2 y} \sum_{\substack{a \leqslant y^{1/5} \\ P^-(a) > w}} \frac{L(a)}{a}.$$

We to replace the sum over a with an unbounded set which is muliplicatively more convenient, starting with

$$\sum_{\substack{a\leqslant y^{1/5}\\ P^-(a)>w}} \frac{L(a)}{a} \geqslant \sum_{\substack{a\leqslant y^{1/5}\\ a\in\mathscr{P}(w,y^{1/15})}} \frac{L(a)}{a} \geqslant \sum_{a\in\mathscr{P}(w,y^{1/15})} \frac{L(a)}{a}\left(1-\frac{\log a}{\log(y^{1/5})}\right).$$

Break this into two sums, the first being what we want and the second involving

$$\sum_{a\in\mathscr{P}(w,y^{1/15})} \frac{L(a)\log a}{a} = \sum_{a\in\mathscr{P}(w,y^{1/15})} \frac{L(a)}{a}\sum_{p|a}\log p = \sum_{w<p\leqslant y^{1/15}} \frac{\log p}{p} \sum_{\substack{b\in\mathscr{P}(w,y^{1/15})\\ p\nmid b}} \frac{L(pb)}{b}.$$

Using the trivial relation $L(pb)\leqslant 2L(b)$ which comes from Lemma 2.1 (ii), and Mertens' estimate, we have

$$\sum_{\substack{a\leqslant y^{1/5}\\ P^-(a)>w}} \frac{L(a)}{a} \geqslant \sum_{a\in\mathscr{P}(w,y^{1/15})} \frac{L(a)}{a}\left(1-\frac{2\log(y^{1/15})+O(1)}{\log(y^{1/5})}\right) \gg \sum_{a\in\mathscr{P}(w,y^{1/15})} \frac{L(a)}{a}.$$

An application of Lemma 2.2 (c) concludes the proof of the lower bound.

For the upper bound, we first relate $H(x,y,2y;\mathcal{R}_w)$ to $H^*(x,y,2y;\mathcal{R}_w)$, the number of *squarefree* integers $n\leqslant x$ with $P^-(n)>w$ and $\tau(n,y,z)\geqslant 1$. Write $n=n'n''$, where n' is squarefree, n'' is squarefull and $(n',n'')=1$. The number of $n\leqslant x$ with $n''>\log^{10} y$ is

$$\leqslant x\sum_{n''>\log^{10} y} \frac{1}{n''} \ll \frac{x}{\log^5 y}.$$

If $n''\leqslant \log^{10} y$, then for some $f|n''$, n' has a divisor in $(y/f,2y/f]$, hence

$$H(x,y,2y;\mathcal{R}_w) \leqslant \sum_{\substack{n''\leqslant\log^{10} y\\ P^-(n)>w}} \sum_{f|n''} H^*\left(\tfrac{x}{n''},\tfrac{y}{f},\tfrac{2y}{f};\mathcal{R}_w\right) + O\left(\frac{x}{\log^5 y}\right). \tag{3.1}$$

In the sum,

$$y/f\leqslant y\leqslant (x/n'')^{1/2}\log^{10} y\leqslant (x/n'')^{5/9}$$

for large enough w_0. We will show that for $w_0\leqslant y_1\leqslant x_1^{5/9}$,

$$H^*(x_1,y_1,2y_1;\mathcal{R}_w) \ll x_1 \max_{t\geqslant y_1^{3/4}} \frac{1}{\log^2 t}\sum_{a\in\mathscr{P}(w,t)} \frac{L(a)}{a}. \tag{3.2}$$

It follows from (3.2) and (3.1) that

$$\begin{aligned} H(x,y,2y;\mathcal{R}_w) &\ll \sum_{\substack{n''\leqslant\log^{10} y\\ P^-(n)>w}} \frac{x}{n''}\sum_{f|n''} \max_{t\geqslant (y/f)^{3/4}} \frac{1}{\log^2 t}\sum_{a\in\mathscr{P}(w,t)} \frac{L(a)}{a} \\ &\ll x\max_{t\geqslant y^{2/3}} \frac{1}{\log^2 t}\sum_{a\in\mathscr{P}(w,t)} \frac{L(a)}{a} \sum_{\substack{n''\leqslant\log^{10} y\\ P^-(n)>w}} \frac{\tau(n'')}{n''}. \end{aligned}$$

The lemma follows by noting that the inner sum over n'' is $O(1)$ and using the relative estimate in Lemma 2.2 (c) with $s = y^{2/3}$, and finally noting that $\mathscr{P}(w, y^{2/3}) \subseteq \mathscr{P}(w, y)$.

It remains to prove (3.2). The right side is $\gg x_1/\log^2 y_1$ since $L(1) = \log 2$, and hence it suffices to count those $n \in (x_1/\log^2 y_1, x_1]$. We'll count separately those $n \in (x_1/2^{r+1}, x_1/2^r]$ for some integer r, $0 \leqslant r \leqslant 5\log_2 y_1$. Let $\mathscr{A}$ be the set of squarefree integers $n \in (x_1/2^{r+1}, x_1/2^r]$ with a divisor in $(y_1, 2y_1]$. Put $z_1 = 2y_1$, $y_2 = \frac{x_1}{2^{r+2}y_1}$, $z_2 = \frac{x_1}{2^r y_1}$. If $n \in \mathscr{A}$, then $n = m_1 m_2$ with $y_i < m_i \leqslant z_i$ $(i = 1, 2)$. For some $j \in \{1, 2\}$ we have $p = P^+(m_j) < P^+(m_{3-j})$; in particular, p is not the largest prime factor of n. Fixing j, we may write $n = abp$, where $P^+(a) < p < P^-(b)$ and $b > p$. Since $\tau(ap, y_j, z_j) \geqslant 1$, we have $y_j/a \leqslant p \leqslant z_j$. By Lemma 2.3 and the fact that $b > p$, given a and p, the number of choices for b is

$$\ll \frac{x_1}{2^r ap\log p} \leqslant \frac{x_1}{2^r ap \log\max\left(P^+(a), y_j/a\right)},$$

Now a has a divisor in $(y_j/p, z_j/p]$, and thus $\log(y_j/p) \in \mathscr{L}(a)$ or $\log(2y_j/p) \in \mathscr{L}(a)$. Since $\mathscr{L}(a)$ is the disjoint union of intervals of length $\geqslant \log 2$ with total measure $L(a)$, by repeated use of Mertens' estimate we obtain

$$\sum_{\substack{\log(cy_j/p)\in\mathscr{L}(a)\\ p\geqslant P^+(a)}} \frac{1}{p} \ll \frac{L(a)}{\log\max\left(P^+(a), y_j/a\right)} \qquad (c = 1, 2).$$

Since $y_j \geqslant y_1^{4/9}/2^{r+2} \geqslant y_1^{3/4}$, we have that

$$H^*(x, y, 2y; \mathcal{R}_w) \ll \sum_{0\leqslant r\leqslant 5\log_2 y_1} \frac{x_1}{2^r} \sum_{t\in\{4y_1, 4y_2\}} \sum_{a\in\mathscr{P}(w,t)} \frac{L(a)}{a\log^2\left(P^+(a) + t/(4a)\right)}.$$

We have $4y_j \geqslant y_1^{4/5}/2^r \geqslant y_1^{3/4}$ for any j and any r. Also, by [**10**, Lemma 2.2],

$$\sum_{a\in\mathscr{P}(w,t)} \frac{L(a)}{a\log^2\left(t/(4a) + P^+(a)\right)} \ll \frac{1}{\log^2 t} \sum_{a\in\mathscr{P}(w,t)} \frac{L(a)}{a}.$$

Summing over r, we deduce (3.2). □

4. Proof of theorem 1: lower bounds

We first deal with simple cases. Let w_0 be a sufficiently large constant and $\varepsilon > 0$ a sufficiently small constant. Firstly, if $y \leqslant w_0$, then Bertrand's postulate implies that there is a prime $p \in (y, 2y]$ and therefore

$$H(x, y, 2y; \mathcal{R}_w) - H_z(x/2, y, 2y; \mathcal{R}_w) \geqslant \#\{x/2 < n \leqslant x : p|n\} \gg x.$$

Also, if $w \leqslant w_0 < y$ and $w \leqslant y/8$, then

$$H(x, y, 2y; \mathcal{R}_w) - H(x/2, y, 2y; \mathcal{R}_w) \geqslant H(x, y, 2y; \mathcal{R}_{w_0}) - H(x/2, y, 2y; \mathcal{R}_{w_0})$$

and the desired bound follows from the case $w = w_0$. Thirdly, when $y > w_0$ and $y^\varepsilon < w \leqslant y/8$, (2.3) implies

$$H(x,y,2y;\mathcal{R}_w) - H(x/2,y,2y;\mathcal{R}_w) \gg \sum_{y<p\leqslant 2y} \#\{x/2p < n \leqslant x/p : P^-(n) > y/8\}$$
$$\gg \sum_{y<p\leqslant 2y} \frac{x}{p\log y} \gg \frac{x}{\log^2 w}.$$

Here it is crucial that $x/p \geqslant 2y/8$, which follows from $y \leqslant \sqrt{x}$.

From now on, we assume

$$w_0 < w \leqslant y^\varepsilon. \tag{4.1}$$

We begin with the local-to-global estimate for $H(x,y,2y;\mathcal{R}_w)$ given in Lemma 3.1, and relate $L(a)$ to counts of *pairs* of divisors which are close together. Evidently,

$$L(a) \geqslant (\log 2)|\{d|a : \tau(a,d,2d) = 0)\}| \geqslant (\log 2)(\tau(a) - W^*(a)), \tag{4.2}$$

where

$$W^*(a) = |\{d|a, d'|a : d \neq d', d < d' \leqslant 2d\}|.$$

We will apply (4.2) with integers whose prime factors are localized. As in [**6**], partition the primes into sets $D_1, D_2, \dots$, where each D_j consists of the primes in an interval $(\lambda_{j-1}, \lambda_j]$, with $\lambda_j \approx \lambda_{j-1}^2$. More precisely, let $\lambda_0 = 1.9$ and define inductively λ_j for $j \geqslant 1$ as the largest prime so that

$$\sum_{\lambda_{j-1}<p\leqslant\lambda_j} \frac{1}{p} \leqslant \log 2. \tag{4.3}$$

For example, $\lambda_1 = 2$ and $\lambda_2 = 7$. By Mertens' bounds, we have

$$\log_2 \lambda_j - \log_2 \lambda_{j-1} = \log 2 + O(1/\log \lambda_{j-1}),$$

and it follows that for some absolute constant K,

$$2^{j-K} \leqslant \log \lambda_j \leqslant 2^{j+K} \qquad (j \geqslant 0). \tag{4.4}$$

For a vector $\mathbf{b} = (b_1, \dots, b_J)$ of non-negative integers, let $\mathscr{A}(\mathbf{b})$ be the set of square-free integers a composed of exactly b_j prime factors from D_j for each j.

Lemma 4.1. *Assume* $\mathbf{b} = (b_1, \dots, b_{J_2})$, *with* $b_j = 0$ *for* $j < J_1$ *and for* $j > J_2$. *Then*

$$\sum_{a\in\mathscr{A}(\mathbf{b})} \frac{W^*(a)}{a} \ll \frac{(2\log 2)^{b_{J_1}+\cdots+b_{J_2}}}{b_{J_1}!\cdots b_{J_2}!} \sum_{j=J_1}^{J_2} 2^{-j+b_{J_1}+\cdots+b_j}.$$

Proof. Identical to the proof of Lemma 2.3 in [**6**], except that we remove the terms corresponding to $d = d'$. □

We will only consider those intervals $D_j \subseteq (w,y]$, that is, only $J_1 \leqslant j \leqslant J_2$, where

$$\begin{aligned} J_1 &:= \min\{j : \lambda_{j-1} > w\} = \frac{\log_2 w}{\log 2} + O(1), \\ J_2 &:= \max\{j : \lambda_j \leqslant y\} = \frac{\log_2 y}{\log 2} + O(1). \end{aligned} \tag{4.5}$$

By (4.1), $J_2 - J_1 \geqslant 1$.

Put

$$M = \frac{\log_2 w}{20}. \tag{4.6}$$

Let $\mathscr{B}_k$ be the set of vectors $(b_{J_1}, \dots, b_{J_2})$ which satisfy

(a) $b_{J_1} + \cdots + b_{J_2} = k$;

(b) $\sum_{j=J_1}^{J_2} 2^{-j+b_{J_1}+\cdots+b_j} \leqslant 2^{-M}$;

(c) $b_{J_1+i-1} \leqslant M + i^2 \quad (i \geqslant 1)$;

(d) $b_{J_2-i+1} \leqslant M + i^2 \quad (i \geqslant 1)$.

Item (b) ensures that the sum on a in Lemma 4.1 is small, provided that w_0 is sufficiently large. From the definition of J_2, whenever $\mathbf{b} \in \mathscr{B}_k$ and $a \in \mathscr{A}(\mathbf{b})$, we have $a \in \mathscr{P}(w, y)$.

By Lemma 4.1, for any k and any $\mathbf{b} \in \mathscr{B}_k$ we have

$$\sum_{a \in \mathscr{A}(\mathbf{b})} \frac{W^*(a)}{a} \leqslant \frac{1}{10} \frac{(\log 4)^k}{b_{J_1}! \cdots b_{J_2}!}. \tag{4.7}$$

By (4.4), the fact that J_1 is sufficiently large, and $b_j \leqslant (j+1-J_1)^2 + M$, for any k and $\mathbf{b} \in \mathscr{B}_k$ we have

$$\begin{aligned} \sum_{a \in \mathscr{A}(\mathbf{b})} \frac{\tau(a)}{a} &= 2^k \prod_{j=J_1}^{J_2} \frac{1}{b_j!} \Bigg(\sum_{p_1 \in D_j} \frac{1}{p_1} \sum_{\substack{p_2 \in D_j \\ p_2 \neq p_1}} \frac{1}{p_2} \cdots \sum_{\substack{p_{b_j} \in D_j \\ p_{b_j} \notin \{p_1, \dots, p_{b_j - 1}\}}} \frac{1}{p_{b_j}} \Bigg) \\ &\geqslant 2^k \prod_{j=J_1}^{J_2} \frac{1}{b_j!} \left(\log 2 - \frac{b_j}{\lambda_{j-1}} \right)^{b_j} \geqslant \frac{(\log 4)^k}{2 b_{J_1}! \cdots b_{J_2}!}. \end{aligned} \tag{4.8}$$

Combining Lemma 3.1, (4.2), Lemma 4.1, and (4.8), we arrive at

$$H(x, y, 2y; \mathcal{R}_w) - H(x/2, y, 2y; \mathcal{R}_w) \gg \frac{x}{\log^2 y} \sum_{k_1 \leqslant k \leqslant k_2} \sum_{\mathbf{b} \in \mathscr{B}_k} \frac{1}{b_{J_1}! \cdots b_{J_2}!} \tag{4.9}$$

for any $k_1 \leqslant k_2$. We bound the sum on $\mathbf{b}$ using techniques from [**5**].

We let

$$k_2 = \lfloor \min(J_2, (\log 4)(J_2 - J_1)) - 2M \rfloor, \qquad k_1 = \left\lfloor k_2 - \tfrac{1}{2} J_1 \right\rfloor. \tag{4.10}$$

Also define

$$v = J_2 - J_1 + 1, \qquad s = J_1 - 2 - M. \tag{4.11}$$

Setting $g_i = b_{J_1+i-1}$ for $i \geqslant 1$, we have

$$\sum_{i=1}^{v} 2^{-i+g_1+\cdots+g_i} = 2^{J_1-1} f(\mathbf{b}) \leqslant 2^{s+1}.$$

By (c) and (d) in the definition of $\mathscr{B}_k$, $g_i \leqslant M + i^2$ and $g_{v+1-i} \leqslant M + i^2$ for every $i \geqslant 1$. Applying the argument on pages 418–419 in [**5**], it follows that for $k_1 \leqslant k \leqslant k_2$ we have

$$\sum_{\mathbf{b} \in \mathscr{B}_k} \frac{(\log 4)^k}{b_{J_1}! \cdots b_{J_2}!} \gg v^k \mathrm{Vol}(Y_k(s, v)), \tag{4.12}$$

where $Y_k(s,v)$ is the set of $\boldsymbol{\xi}=(\xi_1,\dots,\xi_k)\in\mathbb{R}^k$ satisfying

(i) $0\leqslant \xi_1\leqslant\cdots\leqslant\xi_k<1$;
(ii) For $1\leqslant i\leqslant\sqrt{k-M}$, $\xi_{M+i^2}>i/v$ and $\xi_{k+1-(M+i^2)}<1-i/v$;
(iii) $\sum_{j=1}^k 2^{j-v\xi_j}\leqslant 2^s$.

We now invoke a result from [**5**] concerning the volume of $Y_k(s,v)$.

Lemma 4.2 ([**5**, Lemma 4.9])**.** *Suppose* $v\geqslant 1$, $10M\leqslant k\leqslant 100(v-1)$, $s\geqslant M/2+1$ *and* $0\leqslant k-v\leqslant s-M/3-1$. *Then*

$$\mathrm{Vol}(Y_k(s,v))\gg\frac{k-v+1}{(k+1)!}.$$

If w_0 is large enough (implying that M is sufficiently large) and ε is sufficiently small, then (4.5), (4.6), (4.10) and (4.11) together imply that

$$\begin{aligned}
v&=J_2-J_1+1\geqslant 10,\\
10M&=\frac12\log_2 w\leqslant k_1\leqslant k_2\leqslant(\log 4)(J_2-J_1)<2(v-1),\\
s&\geqslant\log_2 w-M\geqslant M/2+1,\\
k_2-v-s&\leqslant(J_2-2M)-(J_2-1-M)=1-M\leqslant -M/3-1.
\end{aligned}$$

Thus, we see that the hypotheses of Lemma 4.2 are satisfied. Also, whenever $k_1\leqslant k\leqslant k_2$ we have

$$k-v+1\asymp J_1\asymp\log_2 w.$$

Therefore, gathering (4.9), (4.12) and invoking Lemma 4.2, we conclude that

$$H(x,y,2y;\mathcal{R}_w)-H(x/2,y,2y;\mathcal{R}_w)\gg\frac{x}{\log^2 y}\left(\frac{\log_2 w}{\log_2 y}\right)\sum_{k_1\leqslant k\leqslant k_2}\frac{(v\log 4)^k}{k!}.$$

Since $k_2\leqslant v\log 4$, we apply (2.1) to bound the final sum and obtain

$$H(x,y,2y;\mathcal{R}_w)-H(x/2,y,2y;\mathcal{R}_w)\gg\frac{x\delta}{\log^2 y}\frac{(v\log 4)^{k_2}}{k_2!}\min\left(v^{1/2},\frac{v\log 4}{v\log 4-k_2},k_2-k_1+1\right).$$

We have $k_2-k_1\asymp J_1\asymp\log_2 w$ and $v=\frac{\log_2 y-\log_2 w}{\log 2}+O(1)$. In the case $\delta\leqslant 1-\frac{1}{\log 4}$, we have (for small enough ε)

$$k_2=\lfloor(\log 4)(J_2-J_1)-2M\rfloor\geqslant v\log 4-O(1)$$

and thus the minimum above is $\gg v^{1/2}$. In this case, we obtain

$$H(x,y,2y;\mathcal{R}_w)-H(x/2,y,2y;\mathcal{R}_w)\gg\frac{x}{\log^2 y}e^{v\log 4}\gg\frac{x}{\log^2 w},$$

as desired. When $\log_2 w<(1-1/\log 4)\log_2 y$, we have $v\asymp\log_2 y$, $k_2=\lfloor J_2-2M\rfloor$ and hence

$$\frac{v\log 4}{v\log 4-k_2}\gg\frac{1}{1-\frac{1}{\log 4}-\delta}+O(1/\log_2 y).$$

In this case the minimum above is

$$\gg \min\left((\log_2 y)^{1/2}, \frac{1}{1-\frac{1}{\log 4}-\delta}+O(1/\log_2 y)\right) \gg (\log_2 y)^{1/2} B(w,y)$$

and, recalling the definition of $\mathcal{E}$, we have by Stirling's formula

$$\frac{(v\log 4)^{k_2}}{k_2!} \gg \frac{(e(1-\delta))^{k_2}}{\sqrt{\log_2 y}} = \frac{(\log y)^{2-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}}}{\sqrt{\log_2 y}}.$$

This completes the proof of the lower bound in Theorem 1.

5. Proof of Theorem 1: upper bounds

In this section, we prove the upper bound in Theorem 1. We begin with simple cases. If w_0 is fixed and $w \leqslant w_0$, then $H(x,y,2y;\mathcal{R}_w) \leqslant H(x,y,2y)$ and the required bound follows from (1.1). Next, if $\log_2 w \geqslant (1-1/\log 4)\log_2 y$, then by Lemma 2.3,

$$H(x,y,2y;\mathcal{R}_w) \leqslant \sum_{\substack{y<d\leqslant 2y \\ P^-(d)>w}} |\{m \leqslant x/d : P^-(m) > w\}| \ll \sum_{\substack{y<d\leqslant 2y \\ P^-(d)>w}} \frac{x}{d\log w} \ll \frac{x}{\log^2 w},$$

as required.

From now on, we assume that

$$\log w_0 \leqslant (\log y)^{1-1/\log 4}, \tag{5.1}$$

that is, $\delta \leqslant 1-\frac{1}{\log 4}$. We apply Lemma 3.1 and use upper bounds for $L(a)$ from Lemma 2.1. As in [**5**], the sums involving $L(a)$ are bounded in terms of multivariate integrals, which were estimated accurately in [**5, 6**].

5.1. **Case I.** $\frac{1}{10} \leqslant \delta \leqslant 1-\frac{1}{\log 4}$**.** This case is very easy, as we expect no clustering of divisors. Let

$$k_0 = \left\lfloor \frac{\log_2 y}{\log 2} \right\rfloor. \tag{5.2}$$

Beginning with Lemma 3.1, we apply Lemma 2.1 (i) to bound $L(a)$ and then apply Lemma 2.2 parts (a) and (b). We have

$$\begin{aligned} H(x,y,2y;\mathcal{R}_w) &\ll \frac{x}{\log^2 y}\left[\sum_{k\leqslant k_0} 2^k \sum_{\substack{a\in\mathscr{P}(w,y) \\ \omega(a)=k}} \frac{1}{a} + \sum_{k>k_0} \sum_{\substack{a\in\mathscr{P}(w,y) \\ \omega(a)=k}} \frac{\log a}{a}\right] \\ &\ll \frac{x}{\log^2 y}\left[\sum_{k\leqslant k_0} \frac{(2\log_2 y - 2\log_2 w)^k}{k!} + (\log y)\sum_{k\geqslant k_0} \frac{(\log_2 y-\log_2 w)^k}{k!}\right]. \end{aligned}$$

Since $k_0 \geqslant 1.4(\log_2 y - \log_2 w)$, the second sum on the right side is dominated by the single term $k=k_0$ and thus by Stirling's formula we get that

$$\sum_{k\geqslant k_0} \frac{(\log_2 y-\log_2 w)^k}{k!} \ll \frac{(\log_2 y-\log_2 w)^{k_0}}{k_0!} \ll \frac{((e\log 2)(1-\delta))^{k_0}}{(\log_2 y)^{1/2}} \ll \frac{(\log y)^{1-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}}}{(\log_2 y)^{1/2}}.$$

We have $k_0 \leqslant 2\log_2 y - 2\log_2 w$ in the first sum, for which we invoke the bound in (2.1) and obtain, with $\alpha = \log_2 y - \log_2 w$ the bound

$$\begin{aligned}\sum_{k\leqslant k_0} \frac{(2\log_2 y - 2\log_2 w)^k}{k!} &\ll \frac{(2\alpha)^{k_0}}{k_0!} \min\left(\alpha^{1/2}, \frac{\alpha}{\alpha - k_0}\right)\\ &\ll (2e(\log 2)(1-\delta))^{k_0} \min\left(1, (\log_2 y)^{-1/2}((1-\delta)\log 4 - 1)^{-1}\right)\\ &\ll (\log y)^{-\mathcal{E} + \frac{\log(1-\delta)}{\log 2}} B(w,y),\end{aligned}$$

as required for Theorem 1.

5.2. **Case II.** $\delta \leqslant \frac{1}{10}$**.** This case is more delicate, because we expect that typically there will be clustering of the divisors of a, we we must bound the probability of non-clustering.

We cut up the sum in Lemma 3.1 according to $\omega(a)$. Let

$$T_k = \sum_{\substack{a\in\mathscr{P}(w,y)\\ \omega(a)=k}} \frac{L(a)}{a}.$$

We bound T_k in terms of a mutivariate integral, in a manner similar to that in [**6**].

Lemma 5.1. *Suppose w is large,* (5.1) *holds, let*

$$v = \left\lfloor \frac{\log_2 y - \log_2 w}{\log 2} \right\rfloor, \quad u = \left\lfloor \frac{\log_2 w}{\log 2} \right\rfloor$$

and assume that $1 \leqslant k \leqslant 10v$*. Then*

$$T_k \ll (2\log_2 y - 2\log_2 w)^k U_k(v,u),$$

where

$$U_k(v,u) = \int\cdots\int_{0\leqslant\xi_1\leqslant\cdots\leqslant\xi_k\leqslant 1} \min_{0\leqslant j\leqslant k} 2^{-j}(2^{v\xi_1+u} + \cdots + 2^{v\xi_j+u} + 1)\, d\boldsymbol{\xi}.$$

Proof. The proof is the same as the proof of Lemma 3.5 in [**6**], except that we make use of the fact that $P^-(a) > w$. Recall the definition of the sets D_j from Section 4. By (4.4), any prime divisor of a lies in D_j with $u - K - 2 \leqslant j \leqslant v + u + K + 3$. Following the proof of [**6**, Lemma 3.5], in particular using Lemma 2.1 (iii), we have

$$T_k \ll \frac{(2\log 2)^k}{k!} \int_{[u-K-2, v+u+K+4]^k} F(\mathbf{t})\, d\mathbf{t},$$

where, letting $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_k$ be the increasing rearrangement of $t_1, \ldots, t_k$,

$$F(\mathbf{t}) = \min_{0\leqslant j\leqslant k} 2^{-j}(2^{s_1} + \cdots + 2^{s_j} + 1).$$

Observe that $F(\mathbf{t})$ is symmetric in $t_1, \ldots, t_k$. Making the change of variables

$$t_i = u - K - 2 + (v + 2K + 6)\xi_i \qquad (1 \leqslant i \leqslant k)$$

we see that $0 \leqslant \xi_i \leqslant 1$ for each i. Utilizing the summetry of $F(\mathbf{t})$, we find that the multiple integral on the right side equals

$$(v+2K+6)^k k! \int \cdots \int_{0\leqslant \xi_1 \leqslant \cdots \leqslant \xi_k \leqslant 1} \min_{0\leqslant j\leqslant k} 2^{-j}\left(2^{(v+2K+6)\xi_i+u}+\cdots+2^{(v+2K+6)\xi_g+u}+1\right) d\boldsymbol{\xi}.$$

We conclude that

$$T_k(y) \ll ((2\log 2)(v+2K+6))^k U_k(v,u).$$

Lastly, $(v+2K+6)^k \ll v^k$ since $k \leqslant 10v$, and the lemma follows. □

To bound $U_k(u,v)$ we invoke the following estimate from [**5, 6**].

Lemma 5.2 ([**5**, Lemma 13.2],[**6**, Lemma 4.4])**.** *Define*

$$\mathscr{T}(k,v,\gamma) = \{\boldsymbol{\xi} \in \mathbb{R}^k : 0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_k \leqslant 1, 2^{v\xi_1}+\cdots+2^{v\xi_j} \geqslant 2^{j-\gamma}\ (1 \leqslant j \leqslant k)\}.$$

Suppose $k, v, \gamma \in \mathbb{Z}$ with $1 \leqslant k \leqslant 10v$ and $\gamma \geqslant 0$. Set $b = k - v$. Then

$$\mathrm{Vol}(\mathscr{T}(k,v,\gamma)) \ll \frac{Y}{2^{2^{b-\gamma}}(k+1)!}, \qquad Y = \begin{cases} b & \text{if } b \geqslant \gamma+5 \\ (\gamma+5-b)^2(\gamma+1) & \text{if } b < \gamma+5 \end{cases}.$$

Lemma 5.3. *Suppose k, u, v are integers satisfying $1 \leqslant k \leqslant 10v$ and $u \geqslant 1$. Then*

$$U_k(v,u) \ll \frac{u(1+|k-v-u|^2)}{(k+1)!(2^{k-v-u}+1)}.$$

Notice that the bound in Lemma 5.3 undergoes a change of behavior at $k = v+u$.

Proof. Put $b = k - v$. For integers $m \geqslant 0$, consider $\boldsymbol{\xi}$ satisfying

$$2^{-m} \leqslant \min_{0\leqslant j\leqslant k} 2^{-j}\left(2^{v\xi_1+u}+\cdots+2^{v\xi_j+u}+1\right) < 2^{1-m}.$$

For $1 \leqslant j \leqslant k$ we have

$$2^{-j}\left(2^{v\xi_1+u}+\cdots+2^{v\xi_j+u}\right) \geqslant \max(2^{-j}, 2^{-m-u}-2^{-j-u}) \geqslant 2^{-m-u-1},$$

and thus $\boldsymbol{\xi} \in \mathscr{T}(k,v,m+u+1)$. Invoking Lemma 5.2, we find that

$$U_k(v,u) \leqslant \sum_{m\geqslant 0} 2^{1-m}\,\mathrm{Vol}(\mathscr{T}(k,v,m+u+1)) \ll \frac{1}{(k+1)!}\sum_{m\geqslant 0}\frac{2^{-m}Y_m}{2^{2^{b-m-u-1}}},$$

$$Y_m = \begin{cases} b & \text{if } m+u \leqslant b-6 \\ (m+u+6-b)^2(m+u+2) & \text{if } m+u > b-6 \end{cases}.$$

Dividing the sum according to the two cases yields

$$\sum_{m\geqslant 0}\frac{2^{-m}Y_m}{2^{2^{b-u-m-1}}} \ll \sum_{0\leqslant m<b-u-5}\frac{b}{2^m 2^{2^{b-m-u-1}}} + \sum_{m\geqslant \max(0,b-u-5)}\frac{(m+u+6-b)^2(m+u+2)}{2^m}.$$

The proof is completed by noting that if $b \geqslant 6+u$, each sum on the right side is $\ll b2^{u-b}$ and if $b \leqslant 5+u$, the first sum is empty and the second is $\ll (6+u-b)^2 \ll 1+(b-u)^2$. □

Finally, we complete the upper bound in Theorem 1. Let $v = \left\lfloor \frac{\log_2 y - \log_2 w}{\log 2} \right\rfloor$, $u = \left\lfloor \frac{\log_2 w}{\log 2} \right\rfloor$ and define k_0 by (5.2). Note that $k_0 = v + u + O(1)$. We now combine Lemmas 5.1 and 5.3. Since $k_0 > 1.4(\log_2 y - \log_2 w)$, we have

$$\begin{aligned}\sum_{k_0 \leqslant k \leqslant 10k_0} T_k &\ll \sum_{k_0 \leqslant k \leqslant 10k_0} \frac{u(1+(k-k_0)^2)}{(k+1)!2^{k-u-v}} (2\log_2 y - 2\log_2 w)^k \\ &\ll u2^{k_0} \sum_{\ell \geqslant 0} \frac{1+\ell^2}{(k_0+1+\ell)!} (\log_2 y - \log_2 w)^{k_0+\ell} \\ &\ll (\log_2 w) \frac{(2\log_2 y - 2\log_2 w)^{k_0}}{(k_0+1)!}.\end{aligned}$$

Similarly, since $k_0 \leqslant 0.9(2\log_2 y - 2\log_2 w)$, we have

$$\begin{aligned}\sum_{0 \leqslant k < k_0} T_k &\ll 1 + \sum_{1 \leqslant k < k_0} \frac{u(k_0-k)^2(2\log_2 y - 2\log_2 w)^k}{(k+1)!} \\ &\ll 1 + u \sum_{\ell=1}^{k_0-1} \frac{u\ell^2(2\log_2 y - 2\log_2 w)^{k_0-\ell}}{(k_0+1-\ell)!} \\ &\ll (\log_2 w) \frac{(2\log_2 y - 2\log_2 w)^{k_0}}{(k_0+1)!}.\end{aligned}$$

For the large values of k we use the crude bound $L(a) \ll \tau(a)$ from Lemma 2.1 (i), followed by an application of Lemma 2.2 (a). This gives

$$\begin{aligned}\sum_{k \geqslant 10k_0} T_k &\leqslant \sum_{k \geqslant 10k_0} \sum_{\substack{a \in \mathscr{P}(w,y) \\ \omega(a)=k}} \frac{2^k \log 2}{a} \leqslant \sum_{k \geqslant 10k_0} \frac{(2\log_2 y - 2\log_2 w + O(1))^k}{k!} \\ &\ll \frac{(2\log_2 y - 2\log_2 w + O(1))^{10k_0}}{(10k_0)!} \\ &\ll \frac{(2\log_2 y - 2\log_2 w)^{k_0}}{(k_0+1)!}.\end{aligned}$$

combining these three bounds for sums of T_k with Lemma 3.1 and Stirling's formula, we conclude that

$$\begin{aligned}H(x,y,2y;\mathcal{R}_w) &\ll \frac{x}{\log^2 y} (\log_2 w) \frac{(2\log_2 y - 2\log_2 w)^{k_0}}{(k_0+1)!} \\ &\ll \frac{x \log_2 w}{(\log_2 y)^{3/2}} (\log y)^{-\mathcal{E} + \frac{\log(1-\delta)}{\log 2}}\end{aligned}$$

and the proof of the upper bound in Theorem 1 is complete.

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