Math. Proc. Camb. Phil. Soc. (2008), **145**, 1

Printed in the United Kingdom

Generalized Euler constants

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(*Received 19 March 2007; revised 7 July 2007*)

Abstract

We study the distribution of a family $\{\gamma(\mathcal{P})\}$ of generalized Euler constants arising from integers sieved by finite sets of primes $\mathcal{P}$. For $\mathcal{P} = \mathcal{P}_r$, the set of the first r primes, $\gamma(\mathcal{P}_r) \to \exp(-\gamma)$ as $r \to \infty$. Calculations suggest that $\gamma(\mathcal{P}_r)$ is monotonic in r, but we prove it is not. Also, we show a connection between the distribution of $\gamma(\mathcal{P}_r) - \exp(-\gamma)$ and the Riemann hypothesis.

1. *Introduction*

Euler's constant $\gamma = 0.5772156649\ldots$ (also known as the Euler-Mascheroni constant) reflects a subtle multiplicative connection between Lebesgue measure and the counting measure of the positive integers and appears in many contexts in mathematics (see e.g. the recent monograph [**4**]). Here we study a class of analogues involving sieved sets of integers and investigate some possible monotonicities.

As a first example, consider the sum of reciprocals of odd integers up to a point x: we have

$$\sum_{\substack{n \le x \\ n \text{ odd}}} \frac{1}{n} = \sum_{n \le x} \frac{1}{n} - \frac{1}{2} \sum_{n \le x/2} \frac{1}{n} = \frac{1}{2} \log x + \frac{\gamma + \log 2}{2} + o(1),$$

and we take

$$\gamma_1 := \lim_{x \to \infty} \Big\{ \sum_{\substack{n \le x \\ n \text{ odd}}} \frac{1}{n} - \frac{1}{2} \log x \Big\} = \frac{\gamma + \log 2}{2}\,.$$

More generally, if $\mathcal{P}$ represents a finite set of primes, let

$$1_{\mathcal{P}}(n) := \begin{cases} 1, & \text{if } (n, \prod_{p \in \mathcal{P}} p) = 1, \\ 0, & \text{else,} \end{cases} \qquad \text{and} \quad \delta_{\mathcal{P}} := \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} 1_{\mathcal{P}}(n).$$

A simple argument shows that $\delta_{\mathcal{P}} = \prod_{p \in \mathcal{P}} (1 - 1/p)$ and that the generalized Euler constant

$$\gamma(\mathcal{P}) := \lim_{x \to \infty} \Big\{ \sum_{n \le x} \frac{1_{\mathcal{P}}(n)}{n} - \delta_{\mathcal{P}} \log x \Big\}$$

exists. We shall investigate the distribution of values of $\gamma(\mathcal{P})$ for various prime sets $\mathcal{P}$.

We begin by indicating two further representations of $\gamma(\mathcal{P})$. First, a small Abelian argument shows that it is the

† Research of the second author supported by National Science Foundation grant DMS-0555367.

constant term in the Laurent series about 1 of the Dirichlet series

$$\sum_1^\infty 1_{\mathcal{P}}(n)n^{-s} = \zeta(s)\prod_{p\in\mathcal{P}}\big(1-p^{-s}\big),$$

where ζ denotes the Riemann zeta function. That is,

$$\gamma(\mathcal{P}) = \lim_{s\to 1}\Big\{\zeta(s)\prod_{p\in\mathcal{P}}\big(1-p^{-s}\big) - \frac{\delta_{\mathcal{P}}}{s-1}\Big\}. \tag{1·1}$$

For a second representation, take $P=\prod_{p\in\mathcal{P}} p$. We have

$$\begin{aligned}\sum_{n\le x}\frac{1_{\mathcal{P}}(n)}{n} &= \sum_{n\le x}\frac{1}{n}\sum_{d|(n,P)}\mu(d) = \sum_{d|P}\frac{\mu(d)}{d}\sum_{m\le x/d}\frac{1}{m}\\ &= \sum_{d|P}\frac{\mu(d)}{d}\left(\log(x/d)+\gamma+O(d/x)\right)\\ &= \delta_{\mathcal{P}}\log x - \sum_{d|P}\frac{\mu(d)\log d}{d} + \gamma\delta_{\mathcal{P}} + o(1)\end{aligned}$$

as $x\to\infty$, where μ is the Möbius function. If we apply the Dirichlet convolution identity $\mu\log = -\Lambda * \mu$, where Λ is the von Mangoldt function, we find that

$$-\sum_{d|P}\frac{\mu(d)\log d}{d} = \sum_{ab|P}\frac{\Lambda(a)\mu(b)}{ab} = \sum_{p\in\mathcal{P}}\frac{\log p}{p}\sum_{b|P/p}\frac{\mu(b)}{b} = \delta_{\mathcal{P}}\sum_{p\in\mathcal{P}}\frac{\log p}{p-1}.$$

Thus we have

PROPOSITION 1. *Let $\mathcal{P}$ be any finite set of primes. Then*

$$\gamma(\mathcal{P}) = \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)\Big\{\gamma+\sum_{p\in\mathcal{P}}\frac{\log p}{p-1}\Big\}. \tag{1·2}$$

We remark that this formula also can be deduced from (1·1) by an easy manipulation.

It is natural to inquire about the spectrum of values

$$G = \{\gamma(\mathcal{P})\colon \mathcal{P} \text{ is a finite set of primes}\}.$$

In particular, what is $\Gamma := \inf G$? The closure of G is simple to describe in terms of Γ.

PROPOSITION 2. *The set G is dense in* $[\Gamma,\infty)$.

Proof. Suppose $x>\Gamma$ and let $\mathcal{P}$ be a finite set of primes with $\gamma(\mathcal{P})<x$. Put

$$c = \big(x-\gamma(\mathcal{P})\big)\prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)^{-1}.$$

Let y be large and let $\mathcal{P}_y$ be the union of $\mathcal{P}$ and the primes in $(y,e^c y]$. By (1·2), the well-known Mertens estimates, and the prime number theorem,

$$\gamma(\mathcal{P}_y) = \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p}\Big)\frac{\log y}{c+\log y}\Big(1+O\Big(\frac{1}{\log y}\Big)\Big)\Big(\gamma+\sum_{p\in\mathcal{P}}\frac{\log p}{p-1}+c+O(1/\log y)\Big).$$

Therefore, $\lim_{y\to\infty}\gamma(\mathcal{P}_y)=x$ and the proof is complete. □

In case $\mathcal{P}$ consists of the first r primes $\{p_1, \ldots, p_r\}$, we replace $\mathcal{P}$ by r in the preceding notation, and let γ_r represent the generalized Euler constant for the integers sieved by the first r primes. Also define $\gamma_0 = \gamma(\emptyset) = \gamma$. These special values play an important role in our theory of generalized Euler constants.

Table 1. *Some Gamma Values (truncated)*

γ = 0.57721			
γ_1 = 0.63518	γ_{11} = 0.56827	γ_{21} = 0.56513	γ_{31} = 0.56385
γ_2 = 0.60655	γ_{12} = 0.56783	γ_{22} = 0.56495	γ_{32} = 0.56378
γ_3 = 0.59254	γ_{13} = 0.56745	γ_{23} = 0.56477	γ_{33} = 0.56372
γ_4 = 0.58202	γ_{14} = 0.56694	γ_{24} = 0.56462	γ_{34} = 0.56365
γ_5 = 0.57893	γ_{15} = 0.56649	γ_{25} = 0.56454	γ_{35} = 0.56361
γ_6 = 0.57540	γ_{16} = 0.56619	γ_{26} = 0.56445	γ_{36} = 0.56355
γ_7 = 0.57352	γ_{17} = 0.56600	γ_{27} = 0.56433	γ_{37} = 0.56350
γ_8 = 0.57131	γ_{18} = 0.56574	γ_{28} = 0.56420	γ_{38} = 0.56345
γ_9 = 0.56978	γ_{19} = 0.56555	γ_{29} = 0.56406	γ_{39} = 0.56341
γ_{10} = 0.56913	γ_{20} = 0.56537	γ_{30} = 0.56391	γ_{40} = 0.56336
			$e^{-\gamma}$ = 0.5614594835…

The next result will be proved in Section 2.

THEOREM 1. *Let $\mathcal{P}$ be a finite set of primes. For some r, $0 \le r \le \#\mathcal{P}$, we have $\gamma(\mathcal{P}) \ge \gamma_r$. Consequently, $\Gamma = \inf_{r\ge 0} \gamma_r$.*

Applying Mertens' well-known formulas for sums and products of primes to (1·2), we find that

$$\gamma_r \sim \frac{e^{-\gamma}}{\log p_r}\{\log p_r + O(1)\} \sim e^{-\gamma} \quad (r \to \infty)\,. \tag{1·3}$$

In particular, $\Gamma \le e^{-\gamma}$ and G is dense in $[e^{-\gamma}, \infty)$.

Values of γ_r for all r with $p_r \le 10^9$ were computed to high precision using PARI/GP. For all such r, $\gamma_r > e^{-\gamma}$ and $\gamma_{r+1} < \gamma_r$. It is natural to ask if these trends persist. That is, (1) Is the sequence of γ_r's indeed decreasing for all $r \ge 1$? (2) If the γ_r's oscillate, are any of them smaller than $e^{-\gamma}$, i.e., is $\Gamma < e^{-\gamma}$? *We shall show that the answer to (1) is No and the answer to (2) is No or Yes depending on whether the Riemann Hypothesis (RH) is true or false.*

THEOREM 2. *There are infinitely many integers r with $\gamma_{r+1} > \gamma_r$ and infinitely many integers r with $\gamma_{r+1} < \gamma_r$.*

Theorem 2 will be proved in Section 3. There we also argue that the smallest r satisfying $\gamma_{r+1} > \gamma_r$ is probably larger than 10^{215}, and hence no amount of computer calculation (today) would detect this phenomenon. This behavior is closely linked to the classical problem of locating sign changes of $\pi(x) - \mathrm{li}(x)$, where $\pi(x)$ is the number of primes $\le x$ and

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x\right) \frac{dt}{\log t}$$

is Gauss' approximation to $\pi(x)$.

Despite the oscillations, $\{\gamma_r\}$ can be shown (on RH) to approach $e^{-\gamma}$ from above. If RH is false, $\{\gamma_r\}$ assumes values above and below $e^{-\gamma}$ (while converging to this value).

THEOREM 3. *Assume RH. Then $\gamma_r > e^{-\gamma}$ for all $r \ge 0$. Moreover, we have*

$$\gamma_r = e^{-\gamma}\left(1 + \frac{g(p_r)}{\sqrt{p_r}(\log p_r)^2}\right), \tag{1·4}$$

where $1.95 \le g(x) \le 2.05$ for large x.

As we shall see later, $\limsup_{x\to\infty} g(x) > 2$ and $\liminf_{x\to\infty} g(x) < 2$.

THEOREM 4. *Assume RH is false. Then $\gamma_r < e^{-\gamma}$ for infinitely many r. In particular, $\Gamma < e^{-\gamma}$.*

COROLLARY 1. *The Riemann Hypothesis is equivalent to the statement "$\gamma_r > e^{-\gamma}$ for all $r \ge 0$."*

It is relatively easy to find reasonable, unconditional lower bounds on Γ by making use of Theorem 1, Proposition 1 and explicit bounds for counting functions of primes. By Theorem 7 of [**10**], we have

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right) \qquad (x \ge 285).$$

Theorem 6 of [**10**] states that

$$\sum_{p \le x} \frac{\log p}{p} > \log x - \gamma - \sum_p \frac{\log p}{p(p-1)} - \frac{1}{2\log x} \qquad (x > 1).$$

Using Proposition 1 and writing $\frac{1}{p-1} = \frac{1}{p} + \frac{1}{p(p-1)}$, we obtain for $x = p_r \ge 285$ the bound

$$\begin{aligned}\gamma_r &\ge \frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right)\left(\gamma + \sum_{p \le x}\frac{\log p}{p} + \sum_p \frac{\log p}{p(p-1)} - \sum_{p \ge x+1}\frac{\log p}{p(p-1)}\right) \\ &\ge \frac{e^{-\gamma}}{\log x}\left(1 - \frac{1}{2\log^2 x}\right)\left(\log x - \frac{1}{2\log x} - \frac{(x+1)(1+\log x)}{x^2}\right).\end{aligned}$$

In the last step we used

$$\sum_{p \ge x+1} \frac{\log p}{p(p-1)} < \frac{x+1}{x}\int_x^\infty \frac{\log t}{t^2}\,dt = \frac{(x+1)(1+\log x)}{x^2}.$$

By the aforementioned computer calculations, $\gamma_r > e^{-\gamma}$ when $p_r < 10^9$, and for $p_r > 10^9$ the bound given above implies that $\gamma_r \ge 0.56$. Therefore, we have unconditionally

$$\Gamma \ge 0.56.$$

Better lower bounds can be achieved by utilizing longer computer calculations, better bounds for prime counts [**9**], and some of the results from §4 below, especially (4·12).

2. *An extremal property of* $\{\boldsymbol{\gamma_r}\}$

In this section we prove Theorem 1. Starting with an arbitrary finite set $\mathcal{P}$ of primes, we perform a sequence of operations on $\mathcal{P}$, at each step either removing the largest prime from our set or replacing the largest prime with a smaller one. We stop when the resulting set is the first r primes, with $0 \le r \le \#\mathcal{P}$. We make strategic choices of the operations to create a sequence of sets of primes $\mathcal{P}_0 = \mathcal{P}, \mathcal{P}_1, \ldots, \mathcal{P}_k$, where

$$\gamma(\mathcal{P}_0) > \gamma(\mathcal{P}_1) > \cdots > \gamma(\mathcal{P}_k)$$

with $\mathcal{P}_k = \{p_1, p_2, \ldots, p_r\}$, the first r primes.

The method is simple to describe. Let $\mathcal{Q} = \mathcal{P}_j$, which is not equal to any set $\{p_1, p_2, \ldots, p_s\}$, and with largest element t. Let $\mathcal{Q}' = \mathcal{Q}\backslash\{t\}$. If $\gamma(\mathcal{Q}') < \gamma(\mathcal{Q})$, we set $\mathcal{P}_{j+1} = \mathcal{Q}'$. Otherwise, we set $\mathcal{P}_{j+1} = \mathcal{Q}' \cup \{u\}$, where u is the smallest prime not in $\mathcal{Q}$. We have $u < t$ by assumption. It remains to show in the latter case that

$$\gamma(\mathcal{P}_{j+1}) < \gamma(\mathcal{Q}). \tag{2·1}$$

By (1·2), for any prime $v \notin \mathcal{Q}'$,

$$\gamma(\mathcal{Q}' \cup \{v\}) = \gamma(\mathcal{Q}')\left(1 - \frac{1}{v} + \frac{\log v}{vA}\right) =: \gamma(\mathcal{Q}')f(v), \quad A := \gamma + \sum_{p \in \mathcal{Q}'} \frac{\log p}{p-1}. \tag{2·2}$$

Observe that $f(v)$ is strictly increasing for $v < e^{A+1}$ and strictly decreasing for $v > e^{A+1}$, and $\lim_{v\to\infty} f(v) = 1$.

Thus $f(v) > 1$ for $e^{A+1} \le v < \infty$. Since $\gamma(\mathcal{Q}) = \gamma(\mathcal{Q}')f(t) \le \gamma(\mathcal{Q}')$, we have $f(t) \le 1$. It follows that $u < t \le e^{A+1}$ and hence $f(u) < f(t) \le 1$. Another application of (2·2), this time with $v = u$, proves (2·1) and the theorem follows.

3. *The $\boldsymbol{\gamma_r}$'s are not monotone*

Define

$$A(x) := \gamma + \sum_{p \le x} \frac{\log p}{p-1}. \tag{3·1}$$

By (2·2), we have

$$\gamma_{r+1} = \gamma_r \left(1 - \frac{1}{p_{r+1}} + \frac{\log p_{r+1}}{p_{r+1} A(p_r)}\right),$$

thus

$$\gamma_{r+1} \le \gamma_r \quad \Longleftrightarrow \quad A(p_r) \ge \log p_{r+1}. \tag{3·2}$$

THEOREM 5. *We have* $A(x) - \log x = \Omega_{\pm}(x^{-1/2} \log\log\log x)$.

Proof. First introduce

$$\Delta(x) := \sum_{p \le x} \frac{\log p}{p-1} - \sum_{n \le x} \frac{\Lambda(n)}{n} \quad \text{and} \quad \theta(x) := \sum_{p \le x} \log p\,. \tag{3·3}$$

Then

$$\begin{aligned} \Delta(x) &= \sum_{p \le x} (\log p) \sum_{\alpha \ge 1} \frac{1}{p^\alpha} - \sum_{p^\alpha \le x} (\log p) \frac{1}{p^\alpha} \\ &= \sum_{p \le x} (\log p) \sum_{\alpha \ge \lfloor \log x / \log p \rfloor + 1} \frac{1}{p^\alpha} = \sum_{p \le x} \frac{\log p}{p-1} p^{-\lfloor \log x / \log p \rfloor} \ge 0. \end{aligned} \tag{3·4}$$

Since $p^{\lfloor \log x / \log p \rfloor} \ge x/p$, we have

$$\Delta(x) = \sum_{\sqrt{x} < p \le x} \frac{\log p}{p(p-1)} + \sum_{x^{1/3} < p \le x^{1/2}} \frac{\log p}{p^2(p-1)} + O\bigg(\sum_{p \le x^{1/3}} \frac{\log p}{x} \bigg). \tag{3·5}$$

Aside from an error of $O(x^{-1})$, the first sum is

$$\sum_{p > \sqrt{x}} \frac{\log p}{p^2} = -\frac{\theta(\sqrt{x})}{x} + \int_{\sqrt{x}}^{\infty} \frac{2\theta(t)}{t^3}\, dt = x^{-1/2} + O(x^{-1/2} \log^{-3} x),$$

using the bound $|\theta(x) - x| \ll x \log^{-3} x$ which follows from the prime number theorem with a suitable error term. The second sum and error term in (3·5) are each $O(x^{-2/3})$, and we deduce that

$$\Delta(x) = \frac{1}{\sqrt{x}} + O\left(\frac{1}{\sqrt{x} (\log x)^3} \right). \tag{3·6}$$

REMARK 1. *Assuming RH and using the von Koch bound* $|\theta(x) - x| \ll \sqrt{x} \log^2 x$, *we obtain the sharper estimate* $\Delta(x) = x^{-1/2} + O(x^{-2/3})$.

By (3·6),

$$A(x) = \gamma + \sum_{n \le x} \frac{\Lambda(n)}{n} + \frac{1}{\sqrt{x}} + O\left(\frac{1}{\sqrt{x} (\log x)^3} \right). \tag{3·7}$$

To analyze the above sum, introduce

$$R(x) := \sum_{n\le x} \frac{\Lambda(n)}{n} - \log x + \gamma. \tag{3·8}$$

For $\Re s > 0$, we compute the Mellin transform

$$\int_1^\infty x^{-s-1} R(x)\, dx = -\frac{1}{s}\frac{\zeta'}{\zeta}(s+1) - \frac{1}{s^2} + \frac{\gamma}{s}. \tag{3·9}$$

The largest real singularity of the function on the right comes from the trivial zero of $\zeta(s+1)$ at $s = -3$.

Let $\rho = \beta + i\tau$ represent a generic nontrivial zero of $\zeta(s)$ – we avoid use of γ for $\Im\rho$ for obvious reasons. If RH is false, there is a zero $\beta + i\tau$ of ζ with $\beta > 1/2$, and a straightforward application of Landau's Oscillation Theorem ([**1**], Theorem 6.31) gives $R(x) = \Omega_\pm(x^{\beta-1-\varepsilon})$ for every $\varepsilon > 0$. In this case, $A(x) - \log x = \Omega_\pm(x^{\beta-1-\varepsilon})$, which is stronger than the assertion of the theorem.

If RH is true, we may analyze $R(x)$ via the explicit formula

$$R_0(x) := \frac{1}{2}\{R(x^-) + R(x^+)\} = -\sum_\rho \frac{x^{\rho-1}}{\rho-1} + \sum_{n=1}^\infty \frac{1}{2n+1}\, x^{-2n-1}\,, \tag{3·10}$$

where $\sum_\rho$ means $\lim_{T\to\infty}\sum_{|\rho|\le T}$. Equation (3·10) is deduced in a standard way from (3·9) by contour integration, and $\lim_{T\to\infty}\sum_{|\rho|\le T}$ converges boundedly for x in any (fixed) compact set contained in $(1, \infty)$. (cf. [**3**], Ch. 17, where a similar formula is given for $\psi_0(x)$, as we now describe.)

In showing that

$$\psi(x) := \sum_{n\le x} \Lambda(n) = x + \Omega_\pm(x^{1/2}\log\log\log x),$$

Littlewood [**6**] (also cf. [**3**], Ch. 17) used the analogous explicit formula

$$\psi_0(x) := \frac{1}{2}\{\psi(x^+) + \psi(x^-)\} = x - \frac{\zeta'}{\zeta}(0) - \sum_\rho \frac{x^\rho}{\rho} + \sum_{n=1}^\infty \frac{1}{2n}\, x^{-2n}$$

and proved that

$$\sum_\rho \frac{x^\rho}{\rho} = \Omega_\pm(\sqrt{x}\log\log\log x).$$

Forming a difference of normalized sums over the non-trivial zeros ρ, we obtain

$$\Big|\sum_\rho \frac{x^{\rho-1/2}}{\rho} - \sum_\rho \frac{x^{\rho-1/2}}{\rho-1}\Big| \ll \sum_\rho \Big|\frac{1}{\rho(\rho-1)}\Big| \ll 1.$$

Thus

$$\sum_\rho \frac{x^{\rho-1}}{\rho-1} = x^{-1/2}\,\Omega_\pm(\log\log\log x),$$

and hence $R(x) = \Omega_\pm(x^{-1/2}\log\log\log x)$.

Therefore, in both cases (RH false, RH true), we have

$$R(x) = \Omega_\pm(x^{-1/2}\log\log\log x),$$

and the theorem follows from (3·7) and (3·8). □

By Theorem 5, there are arbitrarily large values of x for which $A(x) < \log x$. If p_r is the largest prime $\le x$, then

$$A(p_r) = A(x) < \log x < \log p_{r+1}$$

for such x. This implies by (3·2) that $\gamma_{r+1} > \gamma_r$. For the second part of Theorem 2, take x large and satisfying $A(x) > \log x + x^{-1/2}$ and let p_{r+1} be the largest prime $\le x$. By Bertrand's postulate, $p_{r+1} \ge x/2$. Hence

$$A(p_r) = A(p_{r+1}) - \frac{\log p_{r+1}}{p_{r+1}-1} \ge \log x + x^{-1/2} - \frac{\log x}{x/2-1} > \log x \ge \log p_{r+1},$$

which implies $\gamma_{r+1} < \gamma_r$.

Computations with PARI/GP reveal that $\gamma_{r+1} < \gamma_r$ for all r with $p_r < 10^9$. By (3·7), to find r such that $\gamma_{r+1} > \gamma_r$, we need to search for values of x essentially satisfying $R(x) < -x^{-1/2}$. By (3·10), this boils down to finding values of $u = \log x$ such that

$$\sum_{\rho=\beta+i\tau} \frac{e^{iu\tau}}{i\tau - 1/2} > 1.$$

Of course, the smallest zeros of $\zeta(s)$ make the greatest contributions to this sum.

Let $\ell(u)$ be the truncated version of the preceding sum taken over the zeros ρ with $|\Im\rho| \le T_0 := 1132490.66$ (approximately 2 million zeros with positive imaginary part, together with their conjugates). A table of these zeros, accurate to within $3 \cdot 10^{-9}$, is provided on Andrew Odlyzko's web page

`http://www.dtc.umn.edu/~odlyzko/zeta_tables/index.html`.

In computations of $\ell(u)$, the errors in the values of the zeros contribute a total error of at most

$$(3 \cdot 10^{-9})u \sum_{|\Im\rho| \le T_0} \frac{1}{|\rho|} \le (4.5 \cdot 10^{-7})u.$$

Computation using u-values at increments of 10^{-5} and an early abort strategy for u's having too small a sum over the first 1000 zeros, indicates that $\ell(u) \le 0.92$ for $10 \le u \le 495.7$. Thus, it seems likely that the first r with $\gamma_{r+1} > \gamma_r$ occurs when p_r is of size at least $e^{495.7} \approx 1.9 \times 10^{215}$. There is a possibility that the first occurence of $\gamma_{r+1} > \gamma_r$ happens nearby, as $\ell(495.702808) > 0.996$. Going out further, we find that $\ell(1859.129184) > 1.05$, and an averaging method of R. S. Lehman [**5**] can be used to prove that $\gamma_{r+1} > \gamma_r$ for many values of r in the vicinity of $e^{1859.129184} \approx 2.567 \times 10^{807}$. Incidentally, for the problem of locating sign changes of $\pi(x) - \mathrm{li}(x)$, one must find values of u for which (essentially)

$$\sum_{\rho=\beta+i\tau} \frac{e^{iu\tau}}{i\tau + 1/2} < -1.$$

A similarly truncated sum over zeros with $|\Im\rho| \le 600,000$ first attains values less than -1 for positive u values when $u \approx 1.398 \times 10^{316}$ [**2**].

4. *Proof of Theorem* 3

Showing that $\gamma_r > e^{-\gamma}$ for *all* $r \ge 0$ under RH requires explicit estimates for prime numbers. Although sharper estimates are known (cf. [**9**]), older results of Rosser and Schoenfeld suffice for our purposes. The next lemma follows from Theorems 9 and 10 of [**10**].

LEMMA 4·1. *We have* $\theta(x) \le 1.017x$ *for* $x > 0$ *and* $\theta(x) \ge 0.945\,x$ *for* $x \ge 1000$.

The preceeding lemma is unconditional. On RH, we can do better for large x, such as the following results of Schoenfeld ([**11**], Theorem 10 and Corollary 2).

LEMMA 4·2. *Assume RH. Then*

$$|\theta(x) - x| < \frac{\sqrt{x}\log^2 x}{8\pi} \qquad (x \ge 599)$$

and

$$|R(x)| \le \frac{3\log^2 x + 6\log x + 12}{8\pi\sqrt{x}} \qquad (x \ge 8.4).$$

Mertens' formula in the form

$$-\sum_{p\le x} \log(1-1/p) = \log\log x + \gamma + o(1)$$

and a familiar small calculation give

$$-\sum_{p\le x} \log(1-1/p) - \sum_{n\le x} \frac{\Lambda(n)}{n\log n} = \sum_{\substack{p\le x\\ p^a>x}} \frac{1}{ap^a} = O\Big(\frac{1}{\log x}\Big) = o(1).$$

It follows that

$$\sum_{n\le x} \frac{\Lambda(n)}{n\log n} = \log\log x + \gamma + o(1). \tag{4·1}$$

We can obtain an exact expression for the last sum in terms of R (defined in (3·8)) by integrating by parts:

$$\begin{aligned}\sum_{n\le x} \frac{\Lambda(n)}{n\log n} &= \int_{2^-}^{x} \frac{dt/t + dR(t)}{\log t}\\ &= \log\log x - \log\log 2 + \frac{R(x)}{\log x} - \frac{R(2)}{\log 2} + \int_2^x \frac{R(t)\,dt}{t\log^2 t}\\ &= \log\log x + c + \frac{R(x)}{\log x} - \int_x^\infty \frac{R(t)\,dt}{t\log^2 t},\end{aligned}$$

where

$$c := \int_2^\infty \frac{R(t)\,dt}{t\log^2 t} - \frac{R(2)}{\log 2} - \log\log 2 = \gamma,$$

by reference to (4·1) and the relation $R(x) = o(1)$. Thus

$$\sum_{n\le x} \frac{\Lambda(n)}{n\log n} = \log\log x + \gamma + \frac{R(x)}{\log x} - \int_x^\infty \frac{R(t)}{t\log^2 t}\,dt. \tag{4·2}$$

Let $H(x)$ denote the integral in (4·2) and define

$$\widetilde{\Delta}(x) := \sum_{\substack{p\le x\\ p^a>x}} \frac{1}{ap^a}. \tag{4·3}$$

Using (1·2), engaging nearly all the preceding notation and writing $p_r = x$, we have

$$\gamma_r = \frac{e^{-\gamma}}{\log x} \exp\Big\{-\frac{R(x)}{\log x} + H(x) - \widetilde{\Delta}(x)\Big\}\big(\log x + R(x) + \Delta(x)\big). \tag{4·4}$$

We use Lemmas 4·1 and 4·2 to obtain explicit estimates for $H(x),\ \Delta(x)$, and $\widetilde{\Delta}(x)$.

We shall show below that $\Delta(x) - \widetilde{\Delta}(x)\log x \ge 0$. It is crucial for our arguments that this difference be small. Also, although one may use Lemma 4·2 to bound $H(x)$, we shall obtain a much better inequality by using the explicit formula (3·10) for R_0 (which agrees with R a.e.).

LEMMA 4·3. *Assume RH. Then*

$$|H(x)| \le \frac{0.0462}{\sqrt{x}\log^2 x}\Big(1 + \frac{4}{\log x}\Big) \qquad (x \ge 100).$$

Proof. Since $R(x) = o(1)$ by the prime number theorem, we see that the integral defining H converges absolutely. We write

$$H(x) = \lim_{X\to\infty} \int_x^X \frac{R(t)}{t\log^2 t}\,dt,$$

and treat the integral for H as a finite integral in justifying term-wise operations.

We now apply the explicit formula (3·10) for R. For $t \ge 100$,

$$\sum_{n=1}^{\infty} \frac{t^{-2n-1}}{2n+1} \le \frac{0.34}{t^3}$$

and thus

$$\int_x^\infty \frac{1}{t\log^2 t}\sum_{n=1}^{\infty}\frac{t^{-2n-1}}{2n+1}\,dt < \frac{0.12}{x^3\log^2 x} \le \frac{0.0000012}{x^{1/2}\log^2 x}. \tag{4·5}$$

The series over zeta zeros in (3·10) converges boundedly to $R_0(x)$ as $T \to \infty$ for x in a compact region; by the preceding remark on the integral defining H, we can integrate the series term-wise. For each nontrivial zero ρ, integration by parts gives

$$\int_x^\infty \frac{t^{\rho-2}}{\log^2 t}\,dt = \frac{-x^{\rho-1}}{(\rho-1)\log^2 x} + \frac{2}{\rho-1}\int_x^\infty \frac{t^{\rho-2}}{\log^3 t}\,dt$$

and thus

$$\begin{aligned}\left|\int_x^\infty \frac{t^{\rho-2}}{\log^2 t}\,dt\right| &\le \frac{x^{-1/2}}{|\rho-1|\log^2 x} + \frac{2}{|\rho-1|\log^3 x}\int_x^\infty t^{-3/2}\,dt\\ &\le \frac{x^{-1/2}}{|\rho-1|\log^2 x}\left(1+\frac{4}{\log x}\right).\end{aligned}$$

Since RH is assumed true, we have by [**3**], Ch. 12, (10) and (11),

$$\sum_\rho \frac{1}{|\rho-1|^2} = \sum_\rho \frac{1}{|\rho|^2} = 2\sum_\rho \frac{\Re\rho}{|\rho|^2} = 2+\gamma-\log 4\pi = 0.0461914\ldots.$$

Putting these pieces together, we conclude that

$$\begin{aligned}|H(x)| &\le \sum_\rho \frac{1}{|\rho-1|}\left|\int_x^\infty \frac{t^{\rho-2}}{\log^2 t}\,dt\right| + \frac{0.0000012}{\sqrt{x}\log^2 x}\\ &\le \frac{0.0461915+0.0000012}{\sqrt{x}\log^2 x}\left(1+\frac{4}{\log x}\right).\end{aligned}$$

□

Under assumption of RH, we have

$$H(x) = \frac{x^{-1/2}}{\log^2 x}\left\{\sum_\rho \frac{x^{i\tau}}{(\rho-1)^2} + \frac{4\vartheta}{\log x}\sum_\rho \frac{1}{|\rho-1|^2} + \frac{0.12\vartheta'}{x^{5/2}}\right\}, \tag{4·6}$$

where $|\vartheta| \le 1$ and $|\vartheta'| \le 1$. The series are each absolutely summable, and so the first series is an almost periodic function of $\log x$. Thus the values this series assumes are (nearly) repeated infinitely often. The other two terms in (4·6) converge to 0 as $x \to \infty$. Also, the mean value of $H(x)x^{1/2}\log^2 x$ is 0 (integrate the first series); thus the first series in (4·6) assumes both positive and negative values. The $\limsup$ and $\liminf$ of $H(x)x^{1/2}\log^2 x$ are equal to the

$\limsup$ and $\liminf$ of the first series in (4·6), and we have

$$H(x) = \Omega_\pm(x^{-1/2}(\log x)^{-2}).$$

If one assumes that the zeros ρ in the upper half-plane have imaginary parts which are linearly independent over the rationals (unproved even under RH, but widely believed), then Kronecker's theorem implies that

$$\limsup_{x\to\infty} H(x)\sqrt{x}(\log x)^2 = 2+\gamma-\log 4\pi, \quad \liminf_{x\to\infty} H(x)\sqrt{x}(\log x)^2 = -(2+\gamma-\log 4\pi).$$

Continuing to assume RH but making no linear independence assumption on the τ's, we can show that

$$\liminf_{x\to\infty} H(x)\sqrt{x}(\log x)^2 \le -\sum_\rho |\rho-1|^{-2} + \frac{1}{2}\sum_\rho |\rho-1|^{-4} < -0.04615,$$

which is close to $-(2+\gamma-\log 4\pi)$. Indeed, for $x=1$, the first series in (4·6) equals $\sum_\rho(\rho-1)^{-2}$, and by almost periodicity this value is nearly repeated infinitely often. Also,

$$\frac{1}{(\rho-1)^2} + \frac{1}{(\overline{\rho}-1)^2} + \frac{2}{|\rho-1|^2} = \frac{1}{|\rho-1|^4},$$

so that

$$\sum_\rho \frac{1}{(\rho-1)^2} = -\sum_\rho \frac{1}{|\rho-1|^2} + \sum_\rho \frac{1/2}{|\rho-1|^4}.$$

The next two lemmas are unconditional; i.e. they do not depend on RH. We do not try to obtain the sharpest estimates here.

LEMMA 4·4. *We have*

$$\Delta(x) \le \frac{3.05}{\sqrt{x}} \qquad (x \ge 10^6).$$

Proof. Using (3·4) and the upper bound for $\theta(x)$ given in Lemma 4·1,

$$\begin{aligned}\Delta(x) &\le \frac{\sqrt{x}}{\sqrt{x}-1}\sum_{p>\sqrt{x}}\frac{\log p}{p^2} + 2\sum_{p\le\sqrt{x}}\frac{\log p}{x} \\ &\le \frac{1000}{999}\left(-\frac{\theta(\sqrt{x})}{x} + \int_{\sqrt{x}}^\infty \frac{2\theta(t)}{t^3}\,dt\right) + \frac{2\theta(\sqrt{x})}{x} \\ &\le 1.017\left(4 - \frac{1000}{999}\right)x^{-1/2} < 3.05x^{-1/2}.\end{aligned}$$

□

LEMMA 4·5. *We have*

$$\Delta(x)/\log x \ge \widetilde{\Delta}(x) \qquad (x>1) \tag{4·7}$$

$$\frac{\Delta(x)}{\log x} - \widetilde{\Delta}(x) = \frac{2}{\sqrt{x}\log^2 x} + O\left(\frac{1}{\sqrt{x}\log^3 x}\right) \qquad (x \ge 2) \tag{4·8}$$

$$\frac{\Delta(x)}{\log x} - \widetilde{\Delta}(x) \ge \frac{1.23}{\sqrt{x}\log^2 x} \qquad (x \ge 10^6). \tag{4·9}$$

Proof. By (3·4), we have

$$\frac{\Delta(x)}{\log x} - \widetilde{\Delta}(x) = \sum_{p\le x}\sum_{a>\frac{\log x}{\log p}} \frac{1}{p^a}\left(\frac{\log p}{\log x} - \frac{1}{a}\right). \tag{4·10}$$

Each summand on the right side is clearly positive, proving the first part of the lemma.

As shown in the proof of (3·5), the summands of $\Delta(x)$ associated with exponents $a \geq 3$ make a total contribution of $O(x^{-2/3})$. Thus the corresponding summands in (4·10) contribute $O(x^{-2/3}/\log x)$. We handle the remaining term by partial summation, writing

$$\sum_{\sqrt{x}<p\leq x} p^{-2}\left(\frac{\log p}{\log x}-\frac{1}{2}\right) = \int_{\sqrt{x}}^{x} \frac{1}{t^2}\Big\{\frac{1}{\log x}-\frac{1}{2\log t}\Big\}d\theta(t) \tag{4·11}$$

$$= \frac{\theta(x)}{2x^2\log x} + \int_{\sqrt{x}}^{x} \theta(t)t^{-3}\Big\{\frac{2}{\log x}-\frac{1}{\log t}-\frac{1}{2\log^2 t}\Big\}dt.$$

Using the prime number theorem with an error term $\theta(t)-t \ll t\log^{-2} t$, the left side of (4·11) is seen to be

$$= O\left(\frac{1}{\sqrt{x}\log^3 x}\right) + \int_{\sqrt{x}}^{\infty} \Big\{\Big(\frac{2}{t^2\log x}-\frac{1}{t^2\log t}-\frac{1}{t^2\log^2 t}\Big)+\frac{1}{2t^2\log^2 t}\Big\}\,dt$$

$$= O\left(\frac{1}{\sqrt{x}\log^3 x}\right) + \int_{\sqrt{x}}^{\infty} \frac{1}{2t^2\log^2 t}\,dt$$

$$= \frac{2}{\sqrt{x}\log^2 x} + O\left(\frac{1}{\sqrt{x}\log^3 x}\right),$$

proving the second part of the lemma.

The proof of (4·7) shows the expression in (4·11) is a valid lower bound for $\Delta(x)/\log x - \widetilde{\Delta}(x)$. Inserting the estimates from Lemma 4·1 and applying integration by parts gives, for $x \geq 10^6$,

$$\frac{\Delta(x)}{\log x} - \widetilde{\Delta}(x) \geq \frac{0.4725}{x\log x} + 0.945\int_{\sqrt{x}}^{x}\frac{2/\log x - 1/\log t}{t^2}\,dt - \frac{1.017}{2}\int_{\sqrt{x}}^{x}\frac{dt}{t^2\log^2 t}$$

$$= -\frac{0.4725}{x\log x} + 0.4365\int_{\sqrt{x}}^{x}\frac{dt}{t^2\log^2 t}.$$

Another application of integration by parts yields

$$\int_{\sqrt{x}}^{x}\frac{dt}{t^2\log^2 t} = \frac{4}{\sqrt{x}\log^2 x} - \frac{1}{x\log^2 x} - \int_{\sqrt{x}}^{x}\frac{2dt}{t^2\log^3 t}$$

$$\geq \frac{4}{\sqrt{x}\log^2 x} - \frac{1}{x\log^2 x} - \frac{16}{\sqrt{x}\log^3 x}$$

$$\geq \frac{2.84}{\sqrt{x}\log^2 x}.$$

Finally,

$$\frac{1}{x\log x} \leq \frac{\log 10^6}{1000}\frac{1}{\sqrt{x}\log^2 x}.$$

Combining the estimates, we obtain the third part of the lemma. □

We are now set to complete the proof of Theorem 3. A short calculation using PARI/GP verifies that $\gamma_r > e^{-\gamma}$ for $p_r < 10^6$. Assume now that $x = p_r \geq 10^6$. By (4·4),

$$\gamma_r = e^{-\gamma}\left(1+\frac{R(x)+\Delta(x)}{\log x}\right)\exp\left\{-\frac{R(x)+\Delta(x)}{\log x}\right\}\exp\left\{\frac{\Delta(x)}{\log x}-\widetilde{\Delta}(x)+H(x)\right\}. \tag{4·12}$$

By Lemmas 4·2 and 4·4,

$$\frac{|R(x)| + \Delta(x)}{\log x} \le \frac{3\log^2 x + 6\log x + 12 + 24.4\pi}{8\pi\sqrt{x}\log x} \le \frac{0.1556\log x}{\sqrt{x}} \le 0.00215.$$

By Taylor's theorem applied to $-y + \log(1+y)$, if $|y| \le 0.00215$ then $e^{-y}(1+y) \ge e^{-0.501y^2}$. This, together with Lemmas 4·3 and 4·5, yields

$$\gamma_r \ge e^{-\gamma} \exp\left\{-0.01213\frac{\log^2 x}{x} + \frac{1.17}{\sqrt{x}\log^2 x}\right\}.$$

Since $x^{-1/2}\log^4 x$ is decreasing for $x \ge e^8$,

$$\frac{\log^2 x}{x} \le \frac{\log^4 10^6}{1000}\frac{1}{\sqrt{x}\log^2 x} \le \frac{36.431}{\sqrt{x}\log^2 x}.$$

We conclude that

$$\gamma_r \ge e^{-\gamma}\exp\left\{\frac{0.728}{\sqrt{x}\log^2 x}\right\} \qquad (x \ge 10^6),$$

which completes the proof of the first assertion.

By combining (4·12) with Lemma 4·2, Lemma 4·4 and (4·8), we have

$$\gamma_r = e^{-\gamma}\exp\left\{H(x) + \frac{2}{\sqrt{x}\log^2 x} + O\left(\frac{1}{\sqrt{x}\log^3 x}\right)\right\}.$$

Lemma 4·3 implies that

$$|H(x)| \le \frac{0.047}{\sqrt{x}\log^2 x}$$

for large x, and this proves (1·4). By the commentary following the proof of Lemma 4·3, we see that $\liminf g(x) < 2$ and $\limsup g(x) > 2$.

5. *Analysis of $\boldsymbol{\gamma_r}$ if RH is false*

Start with (4·12) and note that $e^{-y}(1+y) \le 1$. Inserting the estimates from Lemma 4·5 gives

$$\gamma_r \le e^{-\gamma}\exp\left\{H(x) + O\left(\frac{1}{\sqrt{x}\log^2 x}\right)\right\}. \tag{5·1}$$

Our goal is to show that $H(x)$ has large oscillations. Basically, a zero of $\zeta(s)$ with real part $\beta > 1/2$ induces oscillations in $H(x)$ of size $x^{\beta-1-\varepsilon}$, which will overwhelm the error term in (5·1).

The Mellin transform of $H(x)$ does not exist because of the blow-up of the integrand near $x = 1$; however the function $H(x)\log x$ is bounded near $x = 1$.

LEMMA 5·1. *For $\Re s > 0$, we have*

$$\int_1^\infty x^{-s-1}H(x)\log x\,dx = -\frac{1}{s^2}\log\left(\frac{s\,\zeta(s+1)}{s+1}\right) - \frac{1-\gamma}{s} + G(s),$$

where $G(s)$ is a function that is analytic for $\Re s > -1$.

Proof. By (4·2),

$$H(x)\log x = -(\log x)\sum_{n\le x}\frac{\Lambda(n)}{n\log n} + R(x) + (\log x)(\log\log x + \gamma).$$

The Mellin transform of the sum is $s^{-1}\log\zeta(s+1)$; hence

$$\int_1^\infty x^{-s-1}(\log x)\sum_{n\le x}\frac{\Lambda(n)}{n\log n}\,dx = -\frac{d}{ds}\frac{\log\zeta(s+1)}{s} = \frac{\log\zeta(s+1)}{s^2} - \frac{1}{s}\frac{\zeta'}{\zeta}(s+1).$$

Let

$$f(x) := \int_1^x \frac{1-t^{-1}}{t\log t}\,dt.$$

We have (cf. (6.7) of [**1**])

$$f(x)\log x = (\log\log x + \gamma)\log x + O\left(\frac{1}{x}\right) \qquad (x>1),$$

and note that a piecewise continuous function which is $O(1/x)$ has a Mellin transform which is analytic for $\Re s > -1$. Also,

$$s\int_1^\infty x^{-s-1}f(x)\,dx = \int_1^\infty x^{-s}f'(x)\,dx = \int_1^\infty x^{-s}\frac{1-x^{-1}}{x\log x}\,dx = \log\left(\frac{s+1}{s}\right).$$

Thus,

$$\int_1^\infty x^{-s-1}f(x)\log x\,dx = -\frac{d}{ds}\frac{1}{s}\log\left(\frac{s+1}{s}\right) = \frac{1}{s^2}\log\left(\frac{s+1}{s}\right) + \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}.$$

Recalling (3·9), the proof is complete. □

We see that the Mellin transform of $H(x)\log x$ has no real singularities in the region $\Re s > -1$. If $\zeta(s)$ has a zero with real part $\beta > 1/2$, Landau's oscillation theorem implies that $H(x)\log x = \Omega_\pm(x^{\beta-1-\varepsilon})$ for every $\varepsilon > 0$. Inequality (5·1) then implies that $\gamma_r < e^{-\gamma}$ for infinitely many r, proving Theorem 4.

REMARK 2. *We leave as an open problem to show that $\gamma_r > e^{-\gamma}$ for infinitely many r in case RH is false. If the supremum σ of real parts of zeros of $\zeta(s)$ is strictly less than 1, then Landau's oscillation theorem immediately gives*

$$H(x) = \Omega_\pm(x^{\sigma-1-\varepsilon})$$

for every $\varepsilon > 0$, while a simpler argument shows that

$$R(x) = O(x^{\sigma-1+\varepsilon}).$$

By (4·4)*, we have*

$$\gamma_r = e^{-\gamma}\exp\left\{H(x) + O\left(\frac{R^2(x)}{\log^2 x}\right) + O\left(\frac{1}{\sqrt{x}\log x}\right)\right\}$$

and the desired result follows immediately. If $\zeta(s)$ has a sequence of zeros with real parts approaching 1, Landau's theorem is too crude to show that $H(x)$ has larger oscillations than does $R^2(x)/\log^2 x$. In this case, techniques of Pintz ([**7**], [**8**]*) are perhaps useful.*

Acknowledement. The authors thank the referee for a careful reading of the paper and for helpful suggestions regarding the exposition.

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