DIOPHANTINE APPROXIMATION WITH ARITHMETIC FUNCTIONS, II

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ABSTRACT. We prove that real numbers can be well-approximated by the normalized Fourier coefficients of newforms.

1. INTRODUCTION

Diophantine approximation results with special sequences shed light on the structure of these sequences. For a variety of results about approximating real numbers by rational numbers with multiplicative restrictions on the denominator, the reader is referred to [3], [4], [10] and the references therein. In this note our objective is to study Diophantine approximation for an important class of multiplicative arithmetical functions arising from modular forms, especially newforms. Recently, the authors studied Diophantine approximation problems for a wide class of additive and multiplicative functions which have regular behavior on primes [2]. Specifically, for any given $\delta > 0$ and $\lambda > 0$, let $\mathcal{F}_{\delta,\lambda}$ be the set of all additive functions $f : \mathbb{N} \to \mathbb{R}$ satisfying the following properties:

(a) We have

$$\sum_{\substack{p \text{ prime} \\ f(p)>0}} f(p) = \infty.$$

(b) There exists a constant $C(f) > 0$ depending on f such that

$$|f(p^v)| \leq \frac{C(f)}{p^\delta},$$

for any prime number p and $v \geq 1$.

(c) There exists $t_0(f) > 0$ depending on f such that for any $0 < t \leq t_0(f)$ there is a prime number p satisfying

$$t - t^{1+\lambda} \leq f(p) \leq t.$$

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We obtained strong simultaneous Diophantine approximation results concerning values of $f \in \mathcal{F}_{\delta,\lambda}$ along various sequences. In particular, our results in [2] can be applied to the additive functions $\log(\frac{\phi(n)}{n})$ and $\log(\frac{\sigma(n)}{n})$ where ϕ is Euler's function and σ is the sum of divisors function. As a consequence, we solved a problem of Erdös [8] about the size of $|\phi(n+1)-\phi(n)|$ and $|\sigma(n+1)-\sigma(n)|$ for infinitely many n and also generalized Diophantine approximation results of Erdös and Schinzel [18], [9] and Wolke [21] to the class $\mathcal{F}_{\delta,\lambda}$. The results from [2] make use of strong results from modern sieve theory.

In this note we investigate Diophantine approximation with normalized coefficients of modular forms using an approach completely different than that of [2]. More specifically, we consider a newform (or Hecke eigenform)

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

with integer coefficients for the congruence subgroup $\Gamma_0(N)$, of even integer weight k, and try to approximate a given real number β using the normalized Fourier coefficients

$$a(n) = \frac{a_f(n)}{n^{\frac{k-1}{2}}}.$$

It is well known that these coefficients are multiplicative, and, by Deligne's work [6], we also know that for any prime p, there exists a unique angle $0 \le \theta_p \le \pi$ such that

$$a(p) = 2\cos\theta_p.$$

Although the behavior of $a(n)$ is irregular, the size of $a(n)$ is very regular on average by the Rankin-Selberg estimate

$$\sum_{n \le x} |a(n)|^2 = A_f x + O(x^{\frac{3}{5}}),$$

where A_f is a constant depending only on f. The distribution of the angles θ_p is predicted by the unproven Sato-Tate conjecture. By showing that there are many primes with $|a(p)|$ a bit larger than 1, strong Ω-type results for $a(n)$ in the case of the full modular group $SL_2(\mathbb{Z})$ have been obtained by Joris [12], Rankin [16], Balasubramanian and Murty [5], Murty [13], and Adhikari [1]. Returning to our Diophantine approximation problem, we prove the following result.

Theorem 1. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

be a newform with integer coefficients for $\Gamma_0(N)$, *of even integer weight* k. *Assume that*

$$\sum_p a(p)^2$$

is divergent, where the summation is over all primes. Then for any real number β*, there exists a positive constant* $C_{f,\beta}$ *depending only on* f *and* β*, such that*

$$|a(n) - \beta| \le \frac{C_{f,\beta}}{\log n}$$

holds for infinitely many positive integers n*.*

We show, for two families of newforms, that $\sum_p a(p)^2$ diverges, and deduce the following corollaries.

Corollary 1. *Let* E *be an elliptic curve over* $\mathbb{Q}$ *with conductor* N *and let*

$$f_E(z) = \sum_{n=1}^{\infty} a_E(n) e^{2\pi i n z}$$

be the weight 2 newform for $\Gamma_0(N)$ *associated to* E*. Then for any real number* β*, there exists a positive constant* $C_{E,\beta}$ *depending only on* E *and* β*, such that*

$$\left| \frac{a_E(n)}{\sqrt{n}} - \beta \right| \le \frac{C_{E,\beta}}{\log n}$$

holds for infinitely many positive integers n*.*

Corollary 2. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

be a newform with integer coefficients for the full modular group $SL_2(\mathbb{Z})$*, of even integer weight* k*. Then for any real number* β*, there exists a positive constant* $C_{f,\beta}$ *depending only on* f *and* β*, such that*

$$|a(n) - \beta| \le \frac{C_{f,\beta}}{\log n}$$

holds for infinitely many positive integers n*.*

A special case of Corollary 2 is of particular interest because of its historical role.

Corollary 3. *Let* $\tau(n)$ *be the Ramanujan Tau function, defined as the* n*th Fourier coefficient of*

$$\Delta(z) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}, \quad q = e^{2\pi i z}.$$

For any real β*, there is a constant* C_β *so that the inequality*

$$\left| \frac{\tau(n)}{n^{11/2}} - \beta \right| \le \frac{C_\beta}{\log n}$$

has infinitely many solutions.

Corollary 3 follows since $\Delta(z)$ is a newform of weight $k = 12$.

Although the distribution of the angles θ_p over primes p is poorly understood and we are nowhere close to being able to prove a property such as (c), for a fixed prime p the behavior of $a(p^r)$ for $r \geq 1$ is very regular. Specifically,

$$a(p^r) = \frac{\sin(r+1)\theta_p}{\sin\theta_p} \tag{1.1}$$

and this relation will play a key role in our argument.

2. Proof of Theorem 1

Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \quad \operatorname{Im} z > 0,$$

be a newform of even weight k, with integer coefficients for $\Gamma_0(N)$. For simplicity, let us denote $a(n) = \frac{a_f(n)}{n^{\frac{k-1}{2}}}$ for any positive integer n. By the work of Deligne [6] on Weil conjectures, we know that for any prime number p,

$$a(p) = \alpha_p + \overline{\alpha_p}$$

for some $\alpha_p \in \mathbb{C}$ with $|\alpha_p| = 1$. Using the recursion

$$a_f(p^{r+1}) = a_f(p)a_f(p^r) - p^{k-1}a_f(p^{r-1})$$

it easily follows by induction on r that

$$a(p^r) = \frac{\alpha_p^{r+1} - \overline{\alpha_p}^{r+1}}{\alpha_p - \overline{\alpha_p}} = \frac{\sin(r+1)\theta_p}{\sin\theta_p},$$

where $\alpha_p = e^{i\theta_p} = \cos\theta_p + i\sin\theta_p$ with $0 \leq \theta_p \leq \pi$. We need information on the set of primes p satisfying the condition that $\frac{\theta_p}{2\pi}$ is irrational. For such p, the numbers $\sin(r+1)\theta_p$ are dense in $[-1, 1]$ and we will exploit this fact in constructing numbers n with $a(n)$ close to β.

Lemma 1. *If $p \geq 5$ and $a_f(p) \neq 0$, then $\frac{\theta_p}{2\pi}$ is irrational.*

Proof. Let p be a prime such that $a_f(p) \neq 0$ and $\frac{\theta_p}{2\pi}$ is rational, so that we may write $\frac{\theta_p}{2\pi} = \frac{A_p}{B_p}$, where $B_p \geq 1$ and $(A_p, B_p) = 1$. Since $\alpha_p = e^{2\pi i \frac{A_p}{B_p}}$ is a primitive B_pth root of unity, we have $|\mathbb{Q}(\alpha_p) : \mathbb{Q}| = \phi(B_p)$. On the other hand, α_p is a zero of the quartic polynomial

$$\begin{aligned} P(z) &= (z - \alpha_p)(z - \overline{\alpha_p})(z + \alpha_p)(z + \overline{\alpha_p}) \\ &= (z^2 - a(p)z + 1)(z^2 + a(p)z + 1) = z^4 + (2 - a^2(p))z^2 + 1. \end{aligned}$$

Since $P(z)$ has rational coefficients, $|\mathbb{Q}(\alpha_p) : \mathbb{Q}| \leq 4$. Hence $\varphi(B_p) \leq 4$, and therefore $B_p \in \{1, 2, 3, 4, 5, 6, 8, 12\}$. We deduce that, if $\frac{\theta_p}{2\pi}$ is rational, then

$$\frac{a_f(p)}{p^{\frac{k-1}{2}}} = a(p) = 2\cos\left(2\pi\frac{A_p}{B_p}\right) \in \left\{0, \pm 1, \pm 2, \pm\sqrt{2}, \pm\sqrt{3}, \pm\left(\frac{\sqrt{5}-1}{2}\right)\right\}.$$

Since $a_f(p)$ is a nonzero integer and k is even, we conclude that $p \in \{2, 3\}$. □

Let $Z_f(x) = |\{p \leq x : a_f(p) = 0\}|$. We remark that, if f has complex multiplication, then by the works of Deuring [7] and Ribet [17], $Z_f(x)$ is asymptotic to $\frac{1}{2}\pi(x)$, and if f does not have complex multiplication, then Serre [20] proved that $Z_f(x) \ll_\varepsilon x(\log x)^{-3/2+\varepsilon}$ for any $\varepsilon > 0$; see also Murty and Murty [14].

Lemma 2. *The product*

$$\prod_{\frac{\theta_p}{2\pi} \notin \mathbb{Q}} \frac{1}{|\sin\theta_p|}$$

diverges.

Proof. For any p with $\theta_p \notin \{0, \pi\}$,

$$\frac{1}{|\sin\theta_p|} = \frac{1}{\sqrt{1-\cos^2\theta_p}} = \left(\frac{1}{1-a(p)^2/4}\right)^{1/2} \geq \exp\left(\frac{a(p)^2}{8}\right).$$

Writing

$$\sum_{p \leq x} a(p)^2 = \sum_{\substack{p \leq x \\ a(p)=0}} a(p)^2 + \sum_{\substack{p \leq x \\ a(p)\neq 0 \\ \frac{\theta_p}{2\pi} \in \mathbb{Q}}} a(p)^2 + \sum_{\substack{p \leq x \\ \frac{\theta_p}{2\pi} \notin \mathbb{Q}}} a(p)^2$$

and observing that the first sum on the right side is zero, the second sum is finite by Lemma 1, and using our assumption that$\sum_p a(p)^2$ is divergent, we obtain that the sum $\sum_{\frac{\theta_p}{2\pi} \notin \mathbb{Q}} a(p)^2$ is divergent. Hence, $\prod_{\frac{\theta_p}{2\pi} \notin \mathbb{Q}} |\sin\theta_p|^{-1}$ also diverges. □

The following is a standard result in Diophantine approximation, see Theorem 10.1 in Chapter 10 of [11].

Lemma 3. *For any irrational α and real λ, there are infinitely many positive integers m such that $\|m\alpha + \lambda\| \leq 3/m$. Here $\|x\|$ denotes the distance from x to the nearest integer.*

Proof of Theorem 1. If p is a prime such that $\frac{\theta_p}{2\pi}$ is irrational, then e.g. by Lemma 3 the numbers

$$a(p^r) = \frac{\sin(r+1)\theta_p}{\sin\theta_p}$$

are dense in the interval $[-\frac{1}{|\sin\theta_p|}, \frac{1}{|\sin\theta_p|}]$. By Lemma 2, given a real number β, there are distinct primes $p_1, p_2, \dots, p_u$ and powers $r_1, \dots, r_u$ with

$$a(p_1^{r_1})a(p_2^{r_2})\cdots a(p_u^{r_u}) > |\beta|.$$

Let q be a prime, distinct from $p_1, \cdots, p_u$, and for which $\frac{\theta_q}{2\pi}$ is irrational. For any $m \geq 1$, define $n_m = p_1^{r_1}\cdots p_u^{r_u}q^{m-1}$, and note that $a(n_m) = a(p_1^{r_1})\cdots a(p_u^{r_u})a(q^{m-1})$. If we let

$$A = \frac{a(p_1^{r_1})\cdots a(p_u^{r_u})}{\sin\theta_q},$$

then $a(n_m) = A\sin(m\theta_q)$ for any $m \geq 1$. By our choice of $p_1^{r_1}, \dots, p_u^{r_u}$ and q, we know that $|A| > |\beta|$, so there is an angle $0 \leq \delta < 2\pi$ such that $\beta = A\sin\delta$. Hence, to approximate β by $a(n_m)$, it is enough to approximate δ by $m\theta_q$ modulo 2π. By Lemma 3, there are infinitely many m for which

$$\left\| m\frac{\theta_q}{2\pi} - \frac{\delta}{2\pi} \right\| \leq \frac{3}{m}.$$

In order to finish the proof, note that

$$|a(n_m) - \beta| = |A|\,|\sin(m\theta_q) - \sin\delta| \leq \frac{6\pi|A|}{m}.$$

On the other hand, using the relation $n_m = p_1^{r_1}\cdots p_u^{r_u}q^{m-1}$ we see that $m \geq c\log n_m$ for some constant $c > 0$ depending on $p_1, \dots, p_u, r_1, \dots, r_u$ and q. Since these numbers are fixed, only depending on β and the newform f, combining the above results we obtain

$$|a(n_m) - \beta| \ll_{f,\beta} \frac{1}{\log n_m},$$

for infinitely many positive integers n_m.

3. Proof of Corollaries 1 and 2

To prove Corollary 1, it suffices to show that

$$\sum_p \frac{a_E(p)^2}{p}$$

is divergent. Note that if this sum were convergent, then the set $\{p : a_E(p) \neq 0\}$ would have density zero in the set of primes and therefore almost all primes would be supersingular primes for E. This is a contradiction, since in the case when E has complex multiplication, the results of Deuring [7] and Ribet [17] show that

$$Z_E(x) = |\{p \leq x : a_E(p) = 0\}|$$

is asymptotic to $\frac{1}{2}\pi(x)$ and if E does not have complex multiplication, then Serre [20] proved that $Z_E(x) = o(\pi(x))$.

Next, we prove Corollary 2. By the works of Joris [12] and Rankin [16] following the classical Rankin-Selberg theory [15], [19] we know that

$$\sum_{p \le x} a(p)^2 \log p$$

is asymptotic to x as $x \to \infty$. It follows easily that

$$\sum_{p \le x} a(p)^2$$

is asymptotic to $\frac{x}{\log x}$ and therefore

$$\sum_{p} a(p)^2$$

is divergent.

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