ON VINOGRADOV'S MEAN VALUE THEOREM: STRONGLY DIAGONAL BEHAVIOUR VIA EFFICIENT CONGRUENCING

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Abstract. We enhance the efficient congruencing method for estimating Vinogradov's integral for moments of order $2s$, with $1 \leqslant s \leqslant k^2-1$. In this way, we prove the main conjecture for such even moments when $1 \leqslant s \leqslant \frac{1}{4}(k+1)^2$, showing that the moments exhibit strongly diagonal behaviour in this range. There are improvements also for larger values of s, these finding application to the asymptotic formula in Waring's problem.

1. Introduction

Considerable progress has recently been achieved in the theory of Vinogradov's mean value theorem (see [12], [14]), associated estimates finding application throughout analytic number theory, in Waring's problem and the theory of the Riemann zeta function, to name but two. The vehicle for these advances is the so-called "efficient congruencing" method, the most striking consequence of which is that the main conjecture in Vinogradov's mean value theorem holds with a number of variables only twice the number conjectured to be best possible (see [12, Theorem 1.1]). Our goal in the present paper is to establish the main conjecture in the complementary variable regime, showing that diagonal behaviour dominates for half of the range conjectured. In common with the previous work cited, this work far exceeds in this direction the conclusions available hitherto for any Diophantine system of large degree k.

When k and s are natural numbers, denote by $J_{s,k}(X)$ the number of integral solutions of the system of Diophantine equations

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant k), \tag{1.1}$$

with $1 \leqslant x_i, y_i \leqslant X$ $(1 \leqslant i \leqslant s)$. The lower bound

$$J_{s,k}(X) \gg X^s + X^{2s-\frac{1}{2}k(k+1)}, \tag{1.2}$$

arises by considering the diagonal solutions of the system (1.1) with $x_i = y_i$ $(1 \leqslant i \leqslant s)$, together with a lower bound for the product of local densities (see [6, equation (7.5)]). Motivated by the latter considerations, the *main*

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conjecture in Vinogradov's mean value theorem asserts that for each $\varepsilon > 0$, one has[1]

$$J_{s,k}(X) \ll X^{\varepsilon}(X^s + X^{2s-\frac{1}{2}k(k+1)}). \tag{1.3}$$

In §7 of this paper, we prove the main conjecture (1.3) for $1 \leqslant s \leqslant \frac{1}{4}(k+1)^2$.

Theorem 1.1. *Suppose that $k \geqslant 4$ and $1 \leqslant s \leqslant \frac{1}{4}(k+1)^2$. Then for each $\varepsilon > 0$, one has*

$$J_{s,k}(X) \ll X^{s+\varepsilon}. \tag{1.4}$$

In the range $1 \leqslant s \leqslant k$, the upper bound $J_{s,k}(X) \ll X^s$ follows directly from the Viéte-Girard-Newton formulae concerning the roots of polynomials. Hitherto, the only other case in which the bound (1.4) had been established was that in which $s = k+1$ (see [3, Lemma 5.4], and [7] for a sharper variant). The extension of the range $1 \leqslant s \leqslant k+1$, in which the bound (1.4) is known to hold, to $1 \leqslant s \leqslant \frac{1}{4}(k+1)^2$ covers half of the total range predicted by the main conjecture. Previous approximations to strongly diagonal behaviour in the range $1 \leqslant s \leqslant \frac{1}{4}(k+1)^2$ were considerably weaker. The second author established that when $s \leqslant k^{3/2}(\log k)^{-1}$, one has the bound

$$J_{s,k}(X) \ll X^{s+\nu_{s,k}+\varepsilon},$$

with $\nu_{s,k} = \exp(-Ak^3/s^2)$, for a certain positive constant A (see [9]), and with $\nu_{s,k} = 4s/k^2$ in the longer range $s \leqslant \frac{1}{4}(k+1)^2$ (see [14]). Both results improve on earlier work of Arkhipov and Karatsuba [1] and Tyrina [5], these authors offering substantially sharper bounds than the classical work of Vinogradov [8] for smaller values of s.

We also improve upon bounds for $J_{s,k}(X)$ given in [12] and [14] in the range $\frac{1}{4}(k+1)^2 < s < k^2 - 1$.

Theorem 1.2. *One has the following upper bounds for $J_{s,k}(X)$.*
(i) Let s and m be non-negative integers with

$$2m \leqslant k \quad \text{and} \quad s \geqslant (k-m)^2 + (k-m).$$

Then for each $\varepsilon > 0$, one has

$$J_{s,k}(X) \ll X^{2s-\frac{1}{2}k(k+1)+\delta_{k,m}+\varepsilon}, \tag{1.5}$$

where

$$\delta_{k,m} = m^2.$$

(ii) Let s and m be non-negative integers with

$$2m \leqslant k-1 \quad \text{and} \quad s \geqslant (k-m)^2 - 1.$$

Then for each $\varepsilon > 0$, one has the upper bound (1.5) with

$$\delta_{k,m} = m^2 + m + \frac{m}{k-m-1}.$$

[1]Throughout this paper, the implicit constant in Vinogradov's notation $\ll$ and $\gg$ may depend on s, k and ε.

We note that the second bound of Theorem 1.2, with $m = 0$, recovers Theorem 1.1 of [14], which asserts that the bound (1.3) holds for $s \geqslant k^2 - 1$. Meanwhile, the first bound of Theorem 1.2, again with $m = 0$, recovers the earlier estimate provided by the main theorem of [12], which delivered (1.3) for $s \geqslant k^2 + k$.

One measure of the strength of Theorem 1.2 compared with previous work is provided by the bound for $J_{s,k}(X)$ furnished in the central case $s = \frac{1}{2}k(k+1)$. For this value of s, it follows from [14, Theorem 1.4] that

$$J_{s,k}(X) \ll X^{s+\Delta},$$

with $\Delta = \frac{1}{8}k^2 + O(k)$. Meanwhile, Theorem 1.2 above establishes such a bound with $\Delta = (\frac{3}{2} - \sqrt{2})k^2 + O(k)$. Note that

$$\tfrac{3}{2} - \sqrt{2} = 0.085786\ldots < 0.125 = \tfrac{1}{8}.$$

More generally, in the situation with $s = \alpha k^2$, in which α is a parameter with $\frac{1}{4} \leqslant \alpha \leqslant 1$, we find from [14, Theorem 1.4] that

$$J_{s,k}(X) \ll X^{2s-\frac{1}{2}k(k+1)+\Delta(\alpha)},$$

where $\Delta(\alpha) = \frac{1}{2}(1-\alpha)^2k^2 + O(k)$. Theorem 1.2, on the other hand, shows that such a bound holds with $\Delta(\alpha) = (1 - \sqrt{\alpha})^2k^2 + O(k)$. Note on this occasion that when $\frac{1}{4} \leqslant \alpha < 1$ one has

$$(1 - \sqrt{\alpha})^2 < \tfrac{1}{2}(1 - \alpha)^2,$$

as is easily verified by a modest computation.

Theorems 1.1 and 1.2 are special cases of a more general estimate, and it is the proof of this which is our focus in §§2 to 7.

Theorem 1.3. *Suppose that k, r and t are positive integers with*

$$k \geqslant 2, \quad \max\{2, \tfrac{1}{2}(k-1)\} \leqslant t \leqslant k, \quad 1 \leqslant r \leqslant k \quad \text{and} \quad r + t \geqslant k. \tag{1.6}$$

Define $\kappa = \kappa(r,t,k)$ by

$$\kappa = r(t+1) - \tfrac{1}{2}(t+r-k)\left(t + r - k - 1 + \frac{2r-2}{t-1}\right). \tag{1.7}$$

Then for each $\varepsilon > 0$, one has

$$J_{r(t+1),k}(X) \ll X^{2r(t+1)-\kappa+\varepsilon}.$$

Theorem 1.1 follows directly from Theorem 1.3 on taking r and t to be suitable integers satisfying $r+t = k$. When k is even we put $r = t = k/2$, and when k is odd we instead put $r = \frac{1}{2}(k+1)$ and $t = \frac{1}{2}(k-1)$. In each case it follows that $s = r(t+1)$ is the largest integer not exceeding $\frac{1}{4}(k+1)^2$, and we have $J_{s,k}(X) \ll X^{s+\varepsilon}$. For smaller values of s, the same conclusion is a consequence of the convexity of exponents that follows from Hölder's inequality[2].

[2]Hölder's inequality was evidently first proved, in a form different from that usually found in textbooks, by L. J. Rogers, *An extension of a certain theorem in inequalities*, Messenger of Math., New Series **XVII** (10) (February 1888), 145–150.

Theorem 1.2 follows in the first case from Theorem 1.3 on putting $r = t = k - m$, since then we obtain

$$\begin{aligned}\kappa(r,t,k) &= (k-m)(k-m+1) - \tfrac{1}{2}(k-2m)(k-2m+1)\\ &= \tfrac{1}{2}k(k+1) - m^2.\end{aligned}$$

Meanwhile, in the second case we put $r = k - m - 1$ and $t = k - m$, in this instance obtaining

$$\begin{aligned}\kappa(r,t,k) &= (k-m-1)(k-m+1) - \tfrac{1}{2}(k-2m-1)\left(k-2m-\frac{2}{k-m-1}\right)\\ &= \tfrac{1}{2}k(k+1) - m^2 - m - \frac{m}{k-m-1}.\end{aligned}$$

In broad strokes, Theorem 1.3 is obtained by fully incorporating the ideas of Arkhipov and Karatsuba [1] and Tyrina [5] into the efficient congruencing method which was first created in [12] and further developed in [14]. The parameters r and t control the way in which solutions of certain systems of congruences are counted (see (3.1) below). The power of the method is enhanced by the flexibility to choose the latter parameters, constrained only by (1.6). In particular, the work in [12] corresponds to the case $r = t = k$, while [14] covers the cases $t = k$ and $r + t = k + 1$. We describe in more detail the role played by r and t in §3. The reader will find the fundamental estimate which lies at the core of our argument in Lemma 3.3 below.

There are consequences of the new estimates supplied by Theorem 1.2 in particular so far as the asymptotic formula in Waring's problem is concerned. By applying the mean value estimates published in work [2] of the first author in combination with mean value estimates restricted to minor arcs established in work [13] of the second author, one may convert improved estimates in Vinogradov's mean value theorem into useful estimates for mean values of exponential sums over kth powers. These in turn lead to improvements in bounds for the number of variables required to establish the anticipated asymptotic formula in Waring's problem. In the present paper we enhance these tools by engineering a hybrid of these approaches, increasing further the improvements stemming from Theorem 1.2. We discuss this new hybrid approach in §8, exploring in §9 consequences for the asymptotic formula in Waring's problem. The details are somewhat complicated, and so we refer the reader to the latter section for a summary of the bounds now available.

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2. Preliminaries

We initiate the proof of Theorem 1.3 by setting up the apparatus necessary for the application of the efficient congruencing method. Here, we take the opportunity to introduce a number of simplifications over the treatments of

[12] and [14] that have become apparent as the method has become more familiar. Since we consider the integer k to be fixed, we abbreviate $J_{s,k}(X)$ to $J_s(X)$ without further comment. Our attention is focused on bounding $J_s(X)$ where, for the moment, we think of s as being an arbitrary natural number. We define the real number λ_s^* by means of the relation

$$\lambda_s^* = \limsup_{X\to\infty} \frac{\log J_s(X)}{\log X}.$$

It follows that, for each $\delta > 0$, and any real number X sufficiently large in terms of s, k and δ, one has $J_s(X) \ll X^{\lambda_s^*+\delta}$. In the language of [12] and [14], the real number λ_s^* is the infimum of the set of exponents λ_s permissible for s and k. In view of the lower bound (1.2), together with a trivial bound for $J_s(X)$, we have

$$\max\{s, 2s - \tfrac{1}{2}k(k+1)\} \leqslant \lambda_s^* \leqslant 2s, \tag{2.1}$$

while the conjectured upper bound (1.3) implies that the first inequality in (2.1) should hold with equality.

Next, we record some conventions that ease our expositary burden in what follows. The letters k, r and t denote fixed positive integers satisfying (1.6), and

$$s = rt.$$

We make sweeping use of vector notation. In particular, we may write $\mathbf{z} \equiv \mathbf{w} \pmod p$ to denote that $z_i \equiv w_i \pmod p$ $(1 \leqslant i \leqslant r)$, $\mathbf{z} \equiv \xi \pmod p$ to denote that $z_i \equiv \xi \pmod p$ $(1 \leqslant i \leqslant r)$, or $[\mathbf{z} \pmod q]$ to denote the r-tuple $(\zeta_1, \dots, \zeta_r)$, where for $1 \leqslant i \leqslant r$ one has $1 \leqslant \zeta_i \leqslant q$ and $z_i \equiv \zeta_i \pmod q$. Also, we employ the convention that whenever $G : [0,1)^k \to \mathbb{C}$ is integrable, then

$$\oint G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} = \int_{[0,1)^k} G(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}.$$

For brevity, we write $\lambda = \lambda_{s+r}^*$. Our goal is to show that $\lambda \leqslant 2(s+r) - \kappa$, in which κ is the carefully chosen target exponent given in (1.7). Let N be an arbitrary natural number, sufficiently large in terms of s, k, t and r, and put

$$\theta = (16t)^{-N-1} \quad \text{and} \quad \delta = (1000Nt^N)^{-1}\theta. \tag{2.2}$$

In view of the definition of λ, there exists a sequence of natural numbers $(X_\ell)_{\ell=1}^\infty$, tending to infinity, with the property that

$$J_{s+r}(X_\ell) > X_\ell^{\lambda-\delta} \quad (\ell \in \mathbb{N}). \tag{2.3}$$

Also, provided that X_ℓ is sufficiently large, one has the corresponding upper bound

$$J_{s+r}(Y) < Y^{\lambda+\delta} \quad \text{for} \quad Y \geqslant X_\ell^{1/2}. \tag{2.4}$$

In the argument that follows, we take a fixed element $X = X_\ell$ of the sequence $(X_\ell)_{\ell=1}^\infty$, which we may assume to be sufficiently large in terms of s, k, r, t and N. We then put $M = X^\theta$. Throughout, constants implied in the notation of Landau and Vinogradov may depend on s, k, r, t, N, θ, and δ, but not on any other variable.

Let p be a fixed prime number with $M < p \leqslant 2M$ to be chosen in due course. That such a prime exists is a consequence of the Prime Number Theorem. When c and ξ are non-negative integers, and $\boldsymbol{\alpha} \in [0,1)^k$, define

$$\mathfrak{f}_c(\boldsymbol{\alpha};\xi) = \sum_{\substack{1\leqslant x\leqslant X\\ x\equiv\xi\ (\mathrm{mod}\ p^c)}} \mathrm{e}(\alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_k x^k), \tag{2.5}$$

where $\mathrm{e}(z)$ denotes the imaginary exponential $\mathrm{e}^{2\pi iz}$. As in [14], we must consider well-conditioned r-tuples of integers belonging to distinct congruence classes modulo a suitable power of p. The following notations are similar to, though slightly simpler than, the corresponding notations introduced in [12] and [14]. Denote by $\Xi_c^r(\xi)$ the set of r-tuples $(\xi_1,\ldots,\xi_r)$, with

$$1 \leqslant \xi_i \leqslant p^{c+1} \quad \text{and} \quad \xi_i \equiv \xi\ (\mathrm{mod}\ p^c) \quad (1 \leqslant i \leqslant r),$$

and such that $\xi_1,\ldots,\xi_r$ are distinct modulo p^{c+1}. We then define

$$\mathfrak{F}_c(\boldsymbol{\alpha};\xi) = \sum_{\boldsymbol{\xi}\in\Xi_c^r(\xi)} \prod_{i=1}^{r} \mathfrak{f}_{c+1}(\boldsymbol{\alpha};\xi_i), \tag{2.6}$$

where the exponential sums $\mathfrak{f}_{c+1}(\boldsymbol{\alpha};\xi_i)$ are defined via (2.5).

Two mixed mean values play leading roles within our arguments. When a and b are positive integers, we define

$$I_{a,b}(X;\xi,\eta) = \oint |\mathfrak{F}_a(\boldsymbol{\alpha};\xi)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s}|\, d\boldsymbol{\alpha} \tag{2.7}$$

and

$$K_{a,b}(X;\xi,\eta) = \oint |\mathfrak{F}_a(\boldsymbol{\alpha};\xi)^2 \mathfrak{F}_b(\boldsymbol{\alpha};\eta)^{2t}|\, d\boldsymbol{\alpha}. \tag{2.8}$$

For future reference, we note that as a consequence of orthogonality, the mean value $I_{a,b}(X;\xi,\eta)$ counts the number of integral solutions of the system

$$\sum_{i=1}^{r}(x_i^j - y_i^j) = \sum_{l=1}^{s}(v_l^j - w_l^j) \quad (1 \leqslant j \leqslant k), \tag{2.9}$$

with

$$1 \leqslant \mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w} \leqslant X, \quad \mathbf{v} \equiv \mathbf{w} \equiv \eta\ (\mathrm{mod}\ p^b),$$
$$[\mathbf{x}\ (\mathrm{mod}\ p^{a+1})] \in \Xi_a^r(\xi) \quad \text{and} \quad [\mathbf{y}\ (\mathrm{mod}\ p^{a+1})] \in \Xi_a^r(\xi).$$

Similarly, the mean value $K_{a,b}(X;\xi,\eta)$ counts the number of integral solutions of the system

$$\sum_{i=1}^{r}(x_i^j - y_i^j) = \sum_{l=1}^{t}\sum_{m=1}^{r}(v_{lm}^j - w_{lm}^j) \quad (1 \leqslant j \leqslant k), \tag{2.10}$$

with

$$1 \leqslant \mathbf{x},\mathbf{y} \leqslant X, \quad [\mathbf{x}\ (\mathrm{mod}\ p^{a+1})] \in \Xi_a^r(\xi), \quad [\mathbf{y}\ (\mathrm{mod}\ p^{a+1})] \in \Xi_a^r(\xi),$$

and for $1 \leqslant l \leqslant t$,

$$1 \leqslant \mathbf{v}_l,\mathbf{w}_l \leqslant X, \quad [\mathbf{v}_l\ (\mathrm{mod}\ p^{b+1})] \in \Xi_b^r(\eta), \quad [\mathbf{w}_l\ (\mathrm{mod}\ p^{b+1})] \in \Xi_b^r(\eta).$$

It is convenient to put

$$I_{a,b}(X) = \max_{1\leqslant \xi\leqslant p^a} \max_{\substack{1\leqslant \eta\leqslant p^b \\ \eta\not\equiv\xi \pmod p}} I_{a,b}(X;\xi,\eta) \tag{2.11}$$

and

$$K_{a,b}(X) = \max_{1\leqslant \xi\leqslant p^a} \max_{\substack{1\leqslant \eta\leqslant p^b \\ \eta\not\equiv\xi \pmod p}} K_{a,b}(X;\xi,\eta). \tag{2.12}$$

Of course, these mean values implicitly depend on our choice of p, and this will depend on s, k, r, t, θ and X_ℓ alone. Since we fix p in the pre-congruencing step described in §6, following the proof of Lemma 6.1, the particular choice will be rendered irrelevant.

The pre-congruencing step requires a definition of $K_{0,b}(X)$ consistent with the conditioning idea, and this we now describe. When $\boldsymbol{\zeta}$ is a tuple of integers, we denote by $\Xi(\boldsymbol{\zeta})$ the set of r-tuples $(\xi_1,\dots,\xi_r) \in \Xi_0^r(0)$ such that $\xi_i \not\equiv \zeta_j \pmod p$ for all i and j. Recalling (2.5), we put

$$\mathfrak{F}(\boldsymbol{\alpha};\boldsymbol{\zeta}) = \sum_{\boldsymbol{\xi}\in\Xi(\boldsymbol{\zeta})} \prod_{i=1}^{r} \mathfrak{f}_1(\boldsymbol{\alpha};\xi_i). \tag{2.13}$$

Finally, we define

$$\widetilde{I}_c(X;\eta) = \oint |\mathfrak{F}(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_c(\boldsymbol{\alpha};\eta)^{2s}|\, d\boldsymbol{\alpha}, \tag{2.14}$$

$$\widetilde{K}_c(X;\eta) = \oint |\mathfrak{F}(\boldsymbol{\alpha};\eta)^2 \mathfrak{F}_c(\boldsymbol{\alpha};\eta)^{2t}|\, d\boldsymbol{\alpha}, \tag{2.15}$$

$$K_{0,c}(X) = \max_{1\leqslant\eta\leqslant p^c} \widetilde{K}_c(X;\eta). \tag{2.16}$$

As in [12] and [14], our arguments are simplified by making transparent the relationship between mean values and their anticipated magnitudes. In this context, we define $[\![J_{s+r}(X)]\!]$ by means of the relation

$$J_{s+r}(X) = X^{2s+2r-\kappa}[\![J_{s+r}(X)]\!]. \tag{2.17}$$

Also, we define $[\![I_{a,b}(X)]\!]$ and $[\![K_{a,b}(X)]\!]$ by means of the relations

$$I_{a,b}(X) = (X/M^b)^{2s}(X/M^a)^{2r-\kappa}[\![I_{a,b}(X)]\!] \tag{2.18}$$

and

$$K_{a,b}(X) = (X/M^b)^{2s}(X/M^a)^{2r-\kappa}[\![K_{a,b}(X)]\!]. \tag{2.19}$$

The lower bound (2.3), in particular, may now be written as

$$[\![J_{s+r}(X)]\!] > X^{\Lambda-\delta}, \tag{2.20}$$

where we have written

$$\Lambda = \lambda - 2(s+r) + \kappa. \tag{2.21}$$

We finish this section by recalling a simple estimate from [12] that encapsulates the translation-dilation invariance of the Diophantine system (1.1).

Lemma 2.1. *Suppose that c is a non-negative integer with $c\theta \leqslant 1$. Then for each natural number u, one has*

$$\max_{1\leqslant \xi\leqslant p^c} \oint |\mathfrak{f}_c(\boldsymbol{\alpha};\xi)|^{2u}\, d\boldsymbol{\alpha} \ll_u J_u(X/M^c).$$

Proof. This is [12, Lemma 3.1]. □

We record an immediate consequence of Lemma 2.1 useful in what follows.

Corollary 2.2. *Suppose that c and d are non-negative integers with $c \leqslant \theta^{-1}$ and $d \leqslant \theta^{-1}$. Then whenever $u, v \in \mathbb{N}$ and $\xi, \zeta \in \mathbb{Z}$, one has*

$$\oint \left|\mathfrak{f}_c(\boldsymbol{\alpha};\xi)^{2u}\mathfrak{f}_d(\boldsymbol{\alpha};\zeta)^{2v}\right| d\boldsymbol{\alpha} \ll_{u,v} \left(J_{u+v}(X/M^c)\right)^{u/(u+v)} \left(J_{u+v}(X/M^d)\right)^{v/(u+v)}.$$

Proof. This follows at once from Lemma 2.1 via Hölder's inequality. □

3. Auxiliary systems of congruences

Following the pattern established in [12], in which efficient congruencing was introduced, and further developed in [14], we begin the main thrust of our analysis with a discussion of the congruences that play a critical role in our method.

Recall the conditions (1.6) on k, r and t. When a and b are integers with $1 \leqslant a < b$, we denote by $\mathcal{B}^r_{a,b}(\mathbf{m};\xi,\eta)$ the set of solutions of the system of congruences

$$\sum_{i=1}^{r}(z_i-\eta)^j \equiv m_j \pmod{p^{jb}} \quad (1\leqslant j\leqslant k), \tag{3.1}$$

with $1 \leqslant \mathbf{z} \leqslant p^{kb}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \pmod{p^{a+1}}$ for some $\boldsymbol{\xi} \in \Xi^r_a(\xi)$. We define an equivalence relation $\mathcal{R}(\lambda)$ on integral r-tuples by declaring the r-tuples $\mathbf{x}$ and $\mathbf{y}$ to be $\mathcal{R}(\lambda)$-equivalent when $\mathbf{x} \equiv \mathbf{y} \pmod{p^\lambda}$. We then write $\mathcal{C}^{r,t}_{a,b}(\mathbf{m};\xi,\eta)$ for the set of $\mathcal{R}(tb)$-equivalence classes of $\mathcal{B}^r_{a,b}(\mathbf{m};\xi,\eta)$, and we define $B^{r,t}_{a,b}(p)$ by putting

$$B^{r,t}_{a,b}(p) = \max_{1\leqslant\xi\leqslant p^a} \max_{\substack{1\leqslant\eta\leqslant p^b\\ \eta\not\equiv\xi \pmod p}} \max_{1\leqslant\mathbf{m}\leqslant p^{kb}} \operatorname{card}(\mathcal{C}^{r,t}_{a,b}(\mathbf{m};\xi,\eta)). \tag{3.2}$$

When $a = 0$ we modify these definitions, so that $\mathcal{B}^r_{0,b}(\mathbf{m};\xi,\eta)$ denotes the set of solutions of the system of congruences (3.1) with $1 \leqslant \mathbf{z} \leqslant p^{kb}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \pmod p$ for some $\boldsymbol{\xi} \in \Xi^r_0(\xi)$, and for which in addition one has $z_i \not\equiv \eta \pmod p$ for $1 \leqslant i \leqslant r$. As in the previous case, we write $\mathcal{C}^{r,t}_{0,b}(\mathbf{m};\xi,\eta)$ for the set of $\mathcal{R}(tb)$-equivalence classes of $\mathcal{B}^r_{0,b}(\mathbf{m};\xi,\eta)$, but we define $B^{r,t}_{0,b}(p)$ by putting

$$B^{r,t}_{0,b}(p) = \max_{1\leqslant\eta\leqslant p^b} \max_{1\leqslant\mathbf{m}\leqslant p^{kb}} \operatorname{card}(\mathcal{C}^{r,t}_{0,b}(\mathbf{m};0,\eta)). \tag{3.3}$$

We note that although the choice of ξ in this situation with $a = 0$ is irrelevant, it is notationally convenient to preserve the similarity with the situation in which $a \geqslant 1$.

Our argument exploits the non-singularity of the solution set underlying $B_{a,b}^{r,t}(p)$ by means of a version of Hensel's lemma made available within the following lemma.

Lemma 3.1. *Let $f_1, \dots, f_d$ be polynomials in $\mathbb{Z}[x_1, \dots, x_d]$ with respective degrees $k_1, \dots, k_d$, and write*

$$J(\mathbf{f}; \mathbf{x}) = \det \left(\frac{\partial f_j}{\partial x_i}(\mathbf{x}) \right)_{1 \leqslant i,j \leqslant d}.$$

When ϖ is a prime number, and l is a natural number, let $\mathcal{N}(\mathbf{f}; \varpi^l)$ denote the number of solutions of the simultaneous congruences

$$f_j(x_1, \dots, x_d) \equiv 0 \pmod{\varpi^l} \quad (1 \leqslant j \leqslant d),$$

with $1 \leqslant x_i \leqslant \varpi^l$ $(1 \leqslant i \leqslant d)$ and $(J(\mathbf{f}; \mathbf{x}), \varpi) = 1$. Then $\mathcal{N}(\mathbf{f}; \varpi^l) \leqslant k_1 \cdots k_d$.

Proof. This is [10, Theorem 1]. □

We recall also an auxiliary lemma from [14], in which terms are eliminated between related polynomial expansions.

Lemma 3.2. *Let α and β be natural numbers. Then there exist integers c_l $(\alpha \leqslant l \leqslant \alpha + \beta)$ and d_m $(\beta \leqslant m \leqslant \alpha + \beta)$, depending at most on α and β, and with $d_\beta \neq 0$, for which one has the polynomial identity*

$$c_\alpha + \sum_{l=1}^{\beta} c_{\alpha+l}(x+1)^{\alpha+l} = \sum_{m=\beta}^{\alpha+\beta} d_m x^m.$$

Proof. This is [14, Lemma 3.2]. □

Our approach to bounding $B_{a,b}^{r,t}(p)$ proceeds by discarding the $k - r$ congruences of smallest modulus p^{jb} $(1 \leqslant j \leqslant k - r)$, but nonetheless aims to lift all solutions to the modulus p^{tb}. The idea of reducing the lifting required, which is tantamount to taking $t < k$, was first exploited by Arkhipov and Karatsuba [1] in the setting of Linnik's classical p-adic approach [4]. Likewise, taking $r < k$ removes from consideration those congruences that require the greatest lifting and produce the biggest inefficiency in the method. Tyrina [5] took $r = t \geqslant k/2$ and further improved bounds on $J_{s,k}(X)$ for $s = O(k^2)$. Later, the second author used a hybrid approach (see [9, Lemma 2.1]), with r and t as free parameters, to obtain large improvements to the bounds for $s = O(k^{3/2-\varepsilon})$.

We also follow a very general approach here, keeping r and t as free parameters, subject only to the necessary constraints given in (1.6). For Theorem 1.1, the crucial observation is that when $r + t = k$, then there is no lifting at all and we capture only diagonal solutions in the symmetric version of (3.1). This observation is reflected in the fact that the coefficients μ and ν imminently to be defined satisfy the condition $\mu = \nu = 0$ in this situation.

The following lemma generalises Lemmata 3.3 to 3.6 of [14]. For future reference, at this point we introduce the coefficients

$$\mu = \tfrac{1}{2}(t+r-k)(t+r-k-1) \quad \text{and} \quad \nu = \tfrac{1}{2}(t+r-k)(k+r-t-1). \tag{3.4}$$

Lemma 3.3. *Suppose that k, r and t satisfy the conditions* (1.6)*, and further that a and b are integers with $0 \leqslant a < b$ and $b \geqslant (k-t-1)a$. Then*

$$B_{a,b}^{r,t}(p) \leqslant k! p^{\mu b + \nu a}.$$

Proof. We suppose in the first instance that $a \geqslant 1$. Fix integers ξ and η with

$$1 \leqslant \xi \leqslant p^a, \quad 1 \leqslant \eta \leqslant p^b \quad \text{and} \quad \eta \not\equiv \xi \pmod{p}.$$

We consider the set of $\mathcal{R}(tb)$-equivalence classes of solutions $\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)$ of the system (3.1), in our first step upgrading a subset of the congruences to the same level. Put

$$\rho = k - r + 1 \quad \text{and} \quad \omega = \max\{0, k-t-1\}.$$

We denote by $\mathcal{D}_1(\mathbf{n})$ the set of $\mathcal{R}(tb)$-equivalence classes of solutions of the system of congruences

$$\sum_{i=1}^{r} (z_i - \eta)^j \equiv n_j \pmod{p^{tb+\omega a}} \quad (\rho \leqslant j \leqslant k), \tag{3.5}$$

with $1 \leqslant \mathbf{z} \leqslant p^{kb}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \pmod{p^{a+1}}$ for some $\boldsymbol{\xi} \in \Xi_a^r(\xi)$.

Recall our assumed bound $b \geqslant \omega a$ and fix an integral k-tuple $\mathbf{m}$. To any solution $\mathbf{z}$ of (3.1) there corresponds a unique r-tuple $\mathbf{n} = (n_\rho, \dots, n_k)$ with $1 \leqslant \mathbf{n} \leqslant p^{tb+\omega a}$ for which (3.5) holds and

$$n_j \equiv m_j \pmod{p^{\sigma(j)}} \quad (\rho \leqslant j \leqslant k),$$

where $\sigma(j) = \min\{jb, tb + \omega a\}$. We therefore infer that

$$\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta) \subseteq \bigcup_{\substack{1 \leqslant n_\rho \leqslant p^{tb+\omega a} \\ n_\rho \equiv m_\rho \pmod{p^{\sigma(\rho)}}}} \cdots \bigcup_{\substack{1 \leqslant n_k \leqslant p^{tb+\omega a} \\ n_k \equiv m_k \pmod{p^{\sigma(k)}}}} \mathcal{D}_1(\mathbf{n}).$$

The number of r-tuples $\mathbf{n}$ in the union is equal to

$$\prod_{j=\rho}^{t} p^{(t-j)b+\omega a} = (p^b)^{\frac{1}{2}(t-\rho)(t-\rho+1)} (p^a)^{(t-\rho+1)\omega} = p^{\mu b + (t-\rho+1)\omega a}.$$

Consequently,

$$\operatorname{card}(\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)) \leqslant p^{\mu b + (t-\rho+1)\omega a} \max_{1 \leqslant \mathbf{n} \leqslant p^{tb+\omega a}} \operatorname{card}(\mathcal{D}_1(\mathbf{n})). \tag{3.6}$$

Observe that for any solution $\mathbf{z}'$ of (3.5) there is an $\mathcal{R}(tb)$-equivalent solution $\mathbf{z}$ satisfying $1 \leqslant \mathbf{z} \leqslant p^{tb+\omega a}$. We next rewrite each variable z_i in the shape $z_i = p^a y_i + \xi$. In view of the hypothesis that $\mathbf{z} \equiv \boldsymbol{\xi} \pmod{p^{a+1}}$ for some $\boldsymbol{\xi} \in \Xi_a^r(\xi)$, the r-tuple $\mathbf{y}$ necessarily satisfies

$$y_i \not\equiv y_m \pmod{p} \quad (1 \leqslant i < m \leqslant r). \tag{3.7}$$

Write $\zeta = \xi - \eta$, and note that the constraint $\eta \not\equiv \xi \pmod{p}$ ensures that $p \nmid \zeta$. We denote the multiplicative inverse of ζ modulo $p^{tb+\omega a}$ by ζ^{-1}. In this way we deduce from (3.5) that $\mathrm{card}(\mathcal{D}_1(\mathbf{n}))$ is bounded above by the number of $\mathcal{R}(tb-a)$-equivalence classes of solutions of the system of congruences

$$\sum_{i=1}^{r}(p^a y_i\zeta^{-1}+1)^j \equiv n_j(\zeta^{-1})^j \pmod{p^{tb+\omega a}} \quad (\rho \leqslant j \leqslant k), \tag{3.8}$$

with $1 \leqslant \mathbf{y} \leqslant p^{tb+(\omega-1)a}$ satisfying (3.7). Let $\mathbf{y} = \mathbf{w}$ be any solution of the system (3.8), if indeed any one such exists. Then we find that all other solutions $\mathbf{y}$ satisfy the system

$$\sum_{i=1}^{r}\big((p^a y_i\zeta^{-1}+1)^j - (p^a w_i\zeta^{-1}+1)^j\big) \equiv 0 \pmod{p^{tb+\omega a}} \quad (\rho \leqslant j \leqslant k). \tag{3.9}$$

Next we make use of Lemma 3.2 just as in the corresponding argument of the proof of [14, Lemmata 3.3 to 3.6]. Consider an index j with $\rho \leqslant j \leqslant k$, and apply the latter lemma with $\alpha = \rho - 1$ and $\beta = j - \rho + 1$. We find that there exist integers $c_{j,l}$ $(\rho - 1 \leqslant l \leqslant j)$ and $d_{j,m}$ $(j-\rho+1 \leqslant m \leqslant j)$, depending at most on j and k, and with $d_{j,j-\rho+1} \neq 0$, for which one has the polynomial identity

$$c_{j,\rho-1} + \sum_{l=\rho}^{j} c_{j,l}(x+1)^l = \sum_{m=j-\rho+1}^{j} d_{j,m}x^m. \tag{3.10}$$

Since we may assume p to be large, moreover, we may suppose that $p \nmid d_{j,j-\rho+1}$. Thus, by multiplying the equation (3.10) through by the multiplicative inverse of $d_{j,j-\rho+1}$ modulo $p^{tb+\omega a}$, we see that there is no loss in supposing that $d_{j,j-\rho+1} \equiv 1 \pmod{p^{tb+\omega a}}$. Taking suitable linear combinations of the congruences comprising (3.9), therefore, we deduce that any solution of this system satisfies

$$(\zeta^{-1}p^a)^{j-\rho+1}\sum_{i=1}^{r}(\psi_j(y_i) - \psi_j(w_i)) \equiv 0 \pmod{p^{tb+\omega a}} \quad (\rho \leqslant j \leqslant k),$$

in which

$$\psi_j(z) = z^{j-\rho+1} + \sum_{m=j-\rho+2}^{j} d_{j,m}(\zeta^{-1}p^a)^{m-j+\rho-1}z^m.$$

We note for future reference that when $a \geqslant 1$, one has

$$\psi_j(z) \equiv z^{j-\rho+1} \pmod{p}. \tag{3.11}$$

Denote by $\mathcal{D}_2(\mathbf{u})$ the set of $\mathcal{R}(tb-a)$-equivalence classes of solutions of the system of congruences

$$\sum_{i=1}^{r}\psi_j(y_i) \equiv u_j \pmod{p^{tb+\omega a-(j-\rho+1)a}} \quad (\rho \leqslant j \leqslant k),$$

with $1 \leqslant \mathbf{y} \leqslant p^{tb+(\omega-1)a}$ satisfying (3.7). Then we have shown thus far that

$$\mathrm{card}(\mathcal{D}_1(\mathbf{n})) \leqslant \max_{1 \leqslant \mathbf{u} \leqslant p^{tb+\omega a}} \mathrm{card}(\mathcal{D}_2(\mathbf{u})). \tag{3.12}$$

Let $\mathcal{D}_3(\mathbf{v})$ denote the set of solutions of the system

$$\sum_{i=1}^{r} \psi_j(y_i) \equiv v_j \ (\mathrm{mod}\ p^{tb-a}) \quad (\rho \leqslant j \leqslant k), \tag{3.13}$$

with $1 \leqslant \mathbf{y} \leqslant p^{tb-a}$ satisfying (3.7). For $\rho \leqslant j \leqslant k$, let

$$\tau(j) = \min\{tb-a, tb+\omega a-(j-\rho+1)a\}.$$

From (1.6) we see that $\tau(k) = tb+\omega a - ra \leqslant tb-a$, and we obtain

$$\begin{aligned} \mathrm{card}(\mathcal{D}_2(\mathbf{u})) &\leqslant \sum_{\substack{1\leqslant v_\rho \leqslant p^{tb-a}\\ v_\rho \equiv u_\rho \ (\mathrm{mod}\ p^{\tau(\rho)})}} \cdots \sum_{\substack{1\leqslant v_k \leqslant p^{tb-a}\\ v_k \equiv u_k \ (\mathrm{mod}\ p^{\tau(k)})}} \mathrm{card}(\mathcal{D}_3(\mathbf{v})) \\ &\leqslant (p^a)^{\frac{1}{2}(r-\omega)(r-\omega-1)} \max_{1\leqslant \mathbf{v}\leqslant p^{tb-a}} \mathrm{card}(\mathcal{D}_3(\mathbf{v})). \end{aligned} \tag{3.14}$$

By combining (3.6), (3.12) and (3.14), we discern at this point that

$$\begin{aligned} \mathrm{card}(\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)) &\leqslant (p^b)^{\mu}(p^a)^{(t-\rho+1)\omega+\frac{1}{2}(r-\omega)(r-\omega-1)} \max_{1\leqslant \mathbf{v}\leqslant p^{tb-a}} \mathrm{card}(\mathcal{D}_3(\mathbf{v})) \\ &= p^{\mu b+\nu a} \max_{1\leqslant \mathbf{v}\leqslant p^{tb-a}} \mathrm{card}(\mathcal{D}_3(\mathbf{v})). \end{aligned} \tag{3.15}$$

It remains now only to bound the number of solutions of the system of congruences (3.13) lying in the set $\mathcal{D}_3(\mathbf{v})$. Define the determinant

$$J(\boldsymbol{\psi};\mathbf{x}) = \det\left(\psi'_{\rho+l-1}(x_i)\right)_{1\leqslant i,l\leqslant r}.$$

In view of (3.11), one has $\psi'_{\rho+l-1}(y_i) \equiv l y_i^{l-1} \ (\mathrm{mod}\ p)$. It follows from (3.7) that

$$\det(y_i^{l-1})_{1\leqslant i,l\leqslant r} = \prod_{1\leqslant i<m\leqslant r} (y_i - y_m) \not\equiv 0 \ (\mathrm{mod}\ p),$$

so that, since $p > k$, we have $(J(\boldsymbol{\psi};\mathbf{y}),p) = 1$. We therefore deduce from Lemma 3.1 that

$$\mathrm{card}(\mathcal{D}_3(\mathbf{v})) \leqslant \rho(\rho+1)\cdots k \leqslant k!,$$

and thus the conclusion of the lemma when $a \geqslant 1$ follows at once from (3.2) and (3.15).

The proof presented above requires only small modifications when $a = 0$. In this case, we denote by $\mathcal{D}_1(\mathbf{n};\eta)$ the set of $\mathcal{R}(tb)$-equivalence classes of solutions of the system of congruences (3.5) with $1 \leqslant \mathbf{z} \leqslant p^{kb}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \ (\mathrm{mod}\ p)$ for some $\boldsymbol{\xi} \in \Xi_0^r(0)$, and for which in addition $z_i \not\equiv \eta \ (\mathrm{mod}\ p)$ for $1 \leqslant i \leqslant r$. Then as in the opening paragraph of our proof, it follows from (3.1) that

$$\mathrm{card}(\mathcal{C}_{0,b}^{r,t}(\mathbf{m};0,\eta)) \leqslant p^{\mu b} \max_{1\leqslant \mathbf{n}\leqslant p^{tb}} \mathrm{card}(\mathcal{D}_1(\mathbf{n};\eta)). \tag{3.16}$$

But $\mathrm{card}(\mathcal{D}_1(\mathbf{n};\eta)) = \mathrm{card}(\mathcal{D}_1(\mathbf{n};0))$, and $\mathrm{card}(\mathcal{D}_1(\mathbf{n};0))$ counts the solutions of the system of congruences

$$\sum_{i=1}^{r} y_i^j \equiv n_j \ (\mathrm{mod}\ p^{tb}) \quad (\rho \leqslant j \leqslant k),$$

with $1 \leqslant \mathbf{y} \leqslant p^{tb}$ satisfying (3.7), and in addition $p \nmid y_i$ $(1 \leqslant i \leqslant r)$. Write

$$J(\mathbf{y}) = \det\big((\rho+j-1)y_i^{\rho+j-2}\big)_{1\leqslant i,j\leqslant r}.$$

Then, since $p > k$, we have

$$J(\mathbf{y}) = \frac{k!}{(\rho-1)!}(y_1\cdots y_r)^{\rho-1}\prod_{1\leqslant i<j\leqslant r}(y_i-y_j) \not\equiv 0 \pmod p.$$

We therefore conclude from Lemma 3.1 that

$$\mathrm{card}(\mathcal{D}_1(\mathbf{n};0)) \leqslant \rho(\rho+1)\cdots k \leqslant k!.$$

In view of (3.3), the conclusion of the lemma therefore follows from (3.16) when $a = 0$. □

4. The conditioning process

We follow the previous treatments of [12] and [14] in seeking next to bound the mean value $I_{a,b}(X;\xi,\eta)$ in terms of analogous mean values $K_{a,b+h}(X;\xi,\zeta)$, in which variables are arranged in "non-singular" blocks. We deviate from these earlier treatments, however, by sacrificing some of the strength of these prior results in order to simplify the proofs. In particular, we are able in this way to avoid introducing coefficient r-tuples from $\{1,-1\}^r$ within the conditioned blocks of variables.

Lemma 4.1. *Let a and b be integers with $b > a \geqslant 1$. Then one has*

$$I_{a,b}(X) \ll K_{a,b}(X) + M^{2s/3}I_{a,b+1}(X).$$

Proof. Fix integers ξ and η with $\eta \not\equiv \xi \pmod p$. Let T_1 denote the number of solutions $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$, $\mathbf{w}$ of the system (2.9) counted by $I_{a,b}(X;\xi,\eta)$ in which $v_1,\dots,v_s$ together occupy at least r distinct residue classes modulo p^{b+1}, and let T_2 denote the corresponding number of solutions in which $v_1,\dots,v_s$ together occupy at most $r-1$ distinct residue classes modulo p^{b+1}. Then

$$I_{a,b}(X;\xi,\eta) = T_1 + T_2. \tag{4.1}$$

We first estimate T_1. Recall the definitions (2.6), (2.7) and (2.8). Then by orthogonality and Hölder's inequality, one finds that

$$\begin{aligned} T_1 &\leqslant \binom{s}{r}\oint |\mathfrak{F}_a(\boldsymbol{\alpha};\xi)|^2\,\mathfrak{F}_b(\boldsymbol{\alpha};\eta)\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{s-r}\mathfrak{f}_b(-\boldsymbol{\alpha};\eta)^s\,d\boldsymbol{\alpha}\\ &\ll (K_{a,b}(X;\xi,\eta))^{1/(2t)}\,(I_{a,b}(X;\xi,\eta))^{1-1/(2t)}. \end{aligned} \tag{4.2}$$

Next, we estimate T_2. In view of the assumptions (1.6), one has $s = rt \geqslant 2r > 2(r-1)$. Consequently, there is an integer $\zeta \equiv \eta \pmod{p^b}$ having the property that at least three of the variables $v_1,\dots,v_s$ are congruent to ζ modulo p^{b+1}.

Hence, again recalling the definitions (2.7) and (2.8), one finds by orthogonality in combination with Hölder's inequality that

$$\begin{aligned} T_2 &\leqslant \binom{s}{3} \sum_{\substack{1\leqslant \zeta\leqslant p^{b+1}\\ \zeta\equiv\eta \pmod{p^b}}} \oint |\mathfrak{F}_a(\boldsymbol{\alpha};\xi)|^2 \mathfrak{f}_{b+1}(\boldsymbol{\alpha};\zeta)^3 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{s-3} \mathfrak{f}_b(-\boldsymbol{\alpha};\eta)^s \, d\boldsymbol{\alpha} \\ &\ll M \left(I_{a,b}(X;\xi,\eta)\right)^{1-3/(2s)} \left(I_{a,b+1}(X)\right)^{3/(2s)}. \end{aligned} \tag{4.3}$$

By substituting (4.2) and (4.3) into (4.1), and recalling (2.11) and (2.12), we therefore conclude that

$$\begin{aligned} I_{a,b}(X) \ll &(K_{a,b}(X))^{1/(2t)} (I_{a,b}(X))^{1-1/(2t)} \\ &+ M (I_{a,b}(X))^{1-3/(2s)} (I_{a,b+1}(X))^{3/(2s)}, \end{aligned}$$

whence

$$I_{a,b}(X) \ll K_{a,b}(X) + M^{2s/3} I_{a,b+1}(X).$$

This completes the proof of the lemma. □

Repeated application of Lemma 4.1, together with a trivial bound for the mean value $K_{a,b+H}(X)$ when H is large enough, yields a relation suitable for iterating the efficient congruencing process.

Lemma 4.2. *Let a and b be integers with $1 \leqslant a < b$, and put $H = 15(b-a)$. Suppose that $b + H \leqslant (2\theta)^{-1}$. Then there exists an integer h with $0 \leqslant h < H$ having the property that*

$$I_{a,b}(X) \ll (M^h)^{2s/3} K_{a,b+h}(X) + (M^H)^{-s/4} (X/M^b)^{2s} (X/M^a)^{\lambda-2s}.$$

Proof. By repeated application of Lemma 4.1, we derive the upper bound

$$I_{a,b}(X) \ll \sum_{h=0}^{H-1} (M^h)^{2s/3} K_{a,b+h}(X) + (M^H)^{2s/3} I_{a,b+H}(X). \tag{4.4}$$

On considering the underlying Diophantine systems, it follows from Corollary 2.2 that

$$\begin{aligned} I_{a,b+H}(X;\xi,\eta) &\leqslant \oint |\mathfrak{f}_a(\boldsymbol{\alpha};\xi)^{2r} \mathfrak{f}_{b+H}(\boldsymbol{\alpha};\eta)^{2s}| \, d\boldsymbol{\alpha} \\ &\ll (J_{s+r}(X/M^a))^{r/(s+r)} (J_{s+r}(X/M^{b+H}))^{s/(s+r)}. \end{aligned}$$

Since $M^{b+H} = (X^\theta)^{b+H} \leqslant X^{1/2}$, we deduce from (2.4) that

$$\begin{aligned} (M^H)^{2s/3} I_{a,b+H}(X) &\ll X^\delta \left((X/M^a)^{r/(s+r)} (X/M^{b+H})^{s/(s+r)}\right)^\lambda (M^H)^{2s/3} \\ &= X^\delta (X/M^b)^{2s} (X/M^a)^{\lambda-2s} M^\Omega, \end{aligned}$$

where

$$\Omega = \lambda \left(a - \frac{ar}{s+r} - \frac{bs}{s+r}\right) + 2s(b-a) + Hs\left(\frac{2}{3} - \frac{\lambda}{s+r}\right).$$

We recall from (2.1) that $\lambda \geqslant s+r$. Then the lower bound $b \geqslant a$ leads to the estimate

$$\Omega \leqslant -s(b-a)\frac{\lambda}{s+r} + 2s(b-a) - \tfrac{1}{3}Hs \leqslant s(b-a) - \tfrac{1}{3}Hs.$$

But $H = 15(b-a)$, and so from (2.2) we discern that

$$\Omega \leqslant -\tfrac{4}{15}Hs \leqslant -\delta\theta^{-1} - \tfrac{1}{4}Hs.$$

We therefore arrive at the estimate

$$(M^H)^{2s/3}I_{a,b+H}(X) \ll (M^H)^{-s/4}(X/M^b)^{2s}(X/M^a)^{\lambda-2s},$$

and the conclusion of the lemma follows on substituting this bound into (4.4). □

5. The efficient congruencing step

We next seek to convert latent congruence information within the mean value $K_{a,b}(X)$ into a form useful in subsequent iterations, this being achieved by using the work of §3. We recall now the definitions of the coefficients μ and ν from (3.4). The following generalises Lemmata 5.1, 5.2, 6.2 and 6.3 of [14].

Lemma 5.1. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$ and $b \geqslant (k-t-1)a$. Then one has*

$$K_{a,b}(X) \ll M^{\mu b+\nu a}(M^{tb-a})^r \left(J_{s+r}(X/M^b)\right)^{1-1/t} \left(I_{b,tb}(X)\right)^{1/t}.$$

Proof. Suppose first that $a \geqslant 1$. Consider fixed integers ξ and η with

$$1 \leqslant \xi \leqslant p^a, \quad 1 \leqslant \eta \leqslant p^b \quad \text{and} \quad \eta \not\equiv \xi \pmod p.$$

The quantity $K_{a,b}(X;\xi,\eta)$ counts integral solutions of the system (2.10) subject to the attendant conditions on $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$, $\mathbf{w}$. As in the argument of the proof of [12, Lemma 6.1], an application of the Binomial Theorem shows that these solutions satisfy the system of congruences

$$\sum_{i=1}^{r}(x_i-\eta)^j \equiv \sum_{i=1}^{r}(y_i-\eta)^j \pmod{p^{jb}} \quad (1 \leqslant j \leqslant k). \tag{5.1}$$

In the notation of §3, it follows that for some k-tuple of integers $\mathbf{m}$, we have $[\mathbf{x} \pmod{p^{kb}}] \in \mathcal{B}^r_{a,b}(\mathbf{m};\xi,\eta)$ and $[\mathbf{y} \pmod{p^{kb}}] \in \mathcal{B}^r_{a,b}(\mathbf{m};\xi,\eta)$. Writing

$$\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\xi,\eta;\mathbf{m}) = \sum_{\boldsymbol{\theta}\in\mathcal{B}^r_{a,b}(\mathbf{m};\xi,\eta)}\prod_{i=1}^{r}\mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i),$$

we see from (2.10) and (5.1) that

$$K_{a,b}(X;\xi,\eta) = \sum_{m_1=1}^{p^b}\dots\sum_{m_k=1}^{p^{kb}}\oint |\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\xi,\eta;\mathbf{m})^2\mathfrak{F}_b(\boldsymbol{\alpha};\eta)^{2t}|\,d\boldsymbol{\alpha}.$$

We now partition the vectors in each set $\mathcal{B}_{a,b}^r(\mathbf{m};\xi,\eta)$ into equivalence classes modulo p^{tb} as in Section 3. An application of Cauchy's inequality leads via Lemma 3.3 to the bound

$$\begin{aligned}|\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\xi,\eta;\mathbf{m})|^2 &= \Big| \sum_{C\in\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)} \sum_{\boldsymbol{\theta}\in C} \prod_{i=1}^r \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2\\ &\leqslant \mathrm{card}(\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)) \sum_{C\in\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)} \Big|\sum_{\boldsymbol{\theta}\in C} \prod_{i=1}^r \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2\\ &\ll M^{\mu b+\nu a} \sum_{C\in\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)} \Big|\sum_{\boldsymbol{\theta}\in C} \prod_{i=1}^r \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2.\end{aligned}$$

Hence

$$K_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} \sum_{\mathbf{m}} \sum_{C\in\mathcal{C}_{a,b}^{r,t}(\mathbf{m};\xi,\eta)} \oint \Big|\sum_{\boldsymbol{\theta}\in C} \prod_{i=1}^r \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2 |\mathfrak{F}_b(\boldsymbol{\alpha};\eta)|^{2t}\,d\boldsymbol{\alpha}.$$

For each k-tuple $\mathbf{m}$ and equivalence class C, the integral above counts solutions of (2.10) with the additional constraints that $[\mathbf{x} \pmod{p^{kb}}] \in C$ and $[\mathbf{y} \pmod{p^{kb}}] \in C$. In particular, $\mathbf{x} \equiv \mathbf{y} \pmod{p^{tb}}$. Moreover, as the sets $\mathcal{B}_{a,b}^r(\mathbf{m};\xi,\eta)$ are disjoint for distinct vectors $\mathbf{m}$ (with $1 \leqslant m_j \leqslant p^{jb}$ for each j), to each pair $(\mathbf{x},\mathbf{y})$ there corresponds at most one pair $(\mathbf{m},C)$. Hence,

$$K_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} H,$$

where H is the number of solutions of (2.10) with the additional hypothesis that $\mathbf{x} \equiv \mathbf{y} \pmod{p^{tb}}$. It follows that

$$K_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} \sum_{\substack{1\leqslant\boldsymbol{\zeta}\leqslant p^{tb}\\ \boldsymbol{\zeta}\equiv\xi \pmod{p^a}}} \oint \Big(\prod_{i=1}^r |\mathfrak{f}_{tb}(\boldsymbol{\alpha};\zeta_i)|^2\Big) |\mathfrak{F}_b(\boldsymbol{\alpha};\eta)|^{2t}\,d\boldsymbol{\alpha}.$$

An application of Hölder's inequality reveals that

$$\begin{aligned}\sum_{\substack{1\leqslant\boldsymbol{\zeta}\leqslant p^{tb}\\ \boldsymbol{\zeta}\equiv\xi \pmod{p^a}}} \prod_{i=1}^r |\mathfrak{f}_{tb}(\boldsymbol{\alpha};\zeta_i)|^2 &= \Big(\sum_{\substack{1\leqslant\zeta\leqslant p^{tb}\\ \zeta\equiv\xi \pmod{p^a}}} |\mathfrak{f}_{tb}(\boldsymbol{\alpha};\zeta)|^2\Big)^r\\ &\leqslant (p^{tb-a})^{r-1} \sum_{\substack{1\leqslant\zeta\leqslant p^{tb}\\ \zeta\equiv\xi \pmod{p^a}}} |\mathfrak{f}_{tb}(\boldsymbol{\alpha};\zeta)|^{2r},\end{aligned}$$

and so it follows that

$$K_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a}(M^{tb-a})^r \max_{\substack{1\leqslant\zeta\leqslant p^{tb}\\ \zeta\equiv\xi \pmod{p^a}}} \oint |\mathfrak{f}_{tb}(\boldsymbol{\alpha};\zeta)^{2r}\mathfrak{F}_b(\boldsymbol{\alpha};\eta)^{2t}|\,d\boldsymbol{\alpha}. \quad (5.2)$$

Next we apply Hölder's inequality to the integral on the right hand side of (5.2) to obtain

$$\oint |\mathfrak{f}_{tb}(\boldsymbol{\alpha};\zeta)^{2r}\mathfrak{F}_b(\boldsymbol{\alpha};\eta)^{2t}|\,d\boldsymbol{\alpha} \leqslant U^{1-1/t}\left(I_{b,tb}(X;\eta,\zeta)\right)^{1/t},$$

where, on considering the underlying Diophantine system and using Lemma 2.1, one has

$$U = \oint |\mathfrak{F}_b(\boldsymbol{\alpha};\eta)|^{2t+2}\,d\boldsymbol{\alpha} \leqslant \oint |\mathfrak{f}_b(\boldsymbol{\alpha};\eta)|^{2s+2r}\,d\boldsymbol{\alpha} \ll J_{s+r}(X/M^b).$$

Notice that since $\eta \not\equiv \xi \pmod p$ and $\zeta \equiv \xi \pmod{p^a}$ with $a \geqslant 1$, one has $\zeta \not\equiv \eta \pmod p$. Then we have $I_{b,tb}(X;\eta,\zeta) \leqslant I_{b,tb}(X)$, and so when $a \geqslant 1$ the conclusion of the lemma follows from (5.2).

When $a = 0$, we must modify the argument slightly. In this case, from (2.15) and (2.16) we find that

$$K_{0,b}(X) = \max_{1\leqslant\eta\leqslant p^b} \oint |\mathfrak{F}(\boldsymbol{\alpha};\eta)^2\mathfrak{F}_b(\boldsymbol{\alpha};\eta)^{2t}|\,d\boldsymbol{\alpha}.$$

The desired conclusion then follows by pursuing the proof given above in the case $a \geqslant 1$, noting that the definition of $\mathfrak{F}(\boldsymbol{\alpha};\eta)$ ensures that the variables resulting from the congruencing argument will avoid the congruence class η modulo p. This completes the proof of the lemma. □

By applying Lemmata 4.2 and 5.1 in tandem, we obtain a sequence of inequalities for the quantities $K_{c,d}(X)$. Recall the definition of Λ from (2.21).

Lemma 5.2. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant (32t\theta)^{-1}$ and $b \geqslant ta$. In addition, put $H = 15(t-1)b$ and $g = b - ta$. Then there exists an integer h, with $0 \leqslant h < H$, having the property that*

$$\begin{aligned}[[K_{a,b}(X)]] \ll X^{\delta}\Big((M^{g-4h/3})^s[[K_{b,tb+h}(X)]]\Big)^{1/t}&(X/M^b)^{\Lambda(1-1/t)}\\ &+ (M^H)^{-r/6}(X/M^b)^{\Lambda}.\end{aligned}$$

Proof. Recall the notational conventions (2.18) and (2.19). The hypotheses $b \geqslant ta$ and (1.6) imply that $b \geqslant (k-t-1)a$. Then it follows from Lemma 5.1 in combination with (2.4) that

$$[[K_{a,b}(X)]] \ll X^{\delta}M^{\omega}[[I_{b,tb}(X)]]^{1/t}(X/M^b)^{\Lambda(1-1/t)}, \tag{5.3}$$

in which we have written

$$\omega = \mu b + \nu a + r(tb-a) + (2r-\kappa)(a-b) - 2s(t-1)b/t.$$

On recalling that $s = rt$ and noting the definition (1.7) of κ, one finds that

$$\begin{aligned}\omega = \kappa(b-a) - (rt - \tfrac{1}{2}(t+r-k)(t+r-k-1))b&\\ + (r + \tfrac{1}{2}(t+r-k)(k+r-t-1))a,&\end{aligned}$$

whence

$$\omega = \left(r - \frac{(t+r-k)(r-1)}{t-1}\right)(b-ta) \leqslant rg.$$

The hypothesized upper bound on b implies that $tb + H \leqslant 16tb \leqslant (2\theta)^{-1}$. We may therefore apply Lemma 4.2 to show that for some integer h with $0 \leqslant h < H$, one has

$$[[I_{b,tb}(X)]] \ll (M^h)^{-4s/3}[[K_{b,tb+h}(X)]] + (M^H)^{-s/4}(X/M^b)^{\Lambda}.$$

We therefore deduce from (5.3) that

$$\begin{aligned}[[K_{a,b}(X)]] \ll{}& X^{\delta}(X/M^b)^{\Lambda(1-1/t)}M^{rg-4rh/3}[[K_{b,tb+h}(X)]]^{1/t}\\ &+ X^{\delta}M^{rg-rH/4}(X/M^b)^{\Lambda}. \end{aligned} \tag{5.4}$$

But in view of the hypotheses (1.6), one has $t \geqslant 2$ and hence

$$H = 15(t-1)b \geqslant 15b \geqslant 15g.$$

Then on recalling (2.2), we find that

$$X^{\delta}(M^r)^{g-H/4} \leqslant M^{\delta\theta^{-1}}(M^{rH})^{1/15-1/4} \leqslant M^{-rH/6}.$$

The conclusion of the lemma therefore follows from (5.4). □

The following crude upper bound for $K_{a,b}(X)$ is a useful addition to our arsenal when b is very large.

Lemma 5.3. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant (2\theta)^{-1}$. Then provided that $\Lambda \geqslant 0$, one has*

$$[[K_{a,b}(X)]] \ll X^{\Lambda+\delta}(M^{b-a})^s.$$

Proof. On considering the underlying Diophantine equations, we deduce from Corollary 2.2 that

$$K_{a,b}(X) \ll (J_{s+r}(X/M^a))^{r/(s+r)}(J_{s+r}(X/M^b))^{s/(s+r)},$$

so that (2.4), (2.17), (2.19) and (2.21) yield the relation

$$\begin{aligned}[[K_{a,b}(X)]] &\ll \frac{X^{\delta}\left((X/M^a)^{r/(s+r)}(X/M^b)^{s/(s+r)}\right)^{2s+2r-\kappa+\Lambda}}{(X/M^b)^{2s}(X/M^a)^{2r-\kappa}}\\ &\leqslant X^{\Lambda+\delta}(M^{b-a})^{\kappa s/(s+r)}.\end{aligned}$$

In view of (1.7), one has $\kappa \leqslant s + r$, and thus the proof of the lemma is complete. □

6. The pre-congruencing step

In order to ensure that the variables in the auxiliary mean values that we consider are appropriately configured, we must expend some additional effort initiating the iteration in a pre-congruencing step. It is at this point that we fix the prime p once and for all. Although we follow the argument of [14, Lemma 6.1] in broad strokes, we are able to obtain some simplification by weakening our conclusions inconsequentially.

Lemma 6.1. *There exists a prime number p with $M < p \leqslant 2M$, and an integer h with $h \in \{0, 1, 2, 3\}$, for which one has*

$$J_{s+r}(X) \ll M^{2s+2sh/3}K_{0,1+h}(X).$$

Proof. The mean value $J_{s+r}(X)$ counts the number of integral solutions of the system

$$\sum_{i=1}^{s+r}(x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant k), \tag{6.1}$$

with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X$. Let T_1 denote the number of these solutions with either two of $x_1, \dots, x_{s+r}$ equal or two of $y_1, \dots, y_{s+r}$ equal, and let T_2 denote the corresponding number of solutions with $x_1, \dots, x_{s+r}$ distinct and $y_1, \dots, y_{s+r}$ distinct. Then we have $J_{s+r}(X) = T_1 + T_2$.

Suppose first that $T_1 \geqslant T_2$. Then by considering the underlying Diophantine systems, it follows from Hölder's inequality that

$$\begin{aligned} J_{s+r}(X) &\leqslant 2T_1 \leqslant 4\binom{s+r}{2}\oint \left|\mathfrak{f}_0(\boldsymbol{\alpha};0)^{2s+2r-2}\mathfrak{f}_0(2\boldsymbol{\alpha};0)\right| d\boldsymbol{\alpha} \\ &\ll \left(\oint |\mathfrak{f}_0(\boldsymbol{\alpha};0)|^{2s+2r}\, d\boldsymbol{\alpha}\right)^{1-1/(s+r)} \left(\oint |\mathfrak{f}_0(2\boldsymbol{\alpha};0)|^{2s+2r}\, d\boldsymbol{\alpha}\right)^{1/(2s+2r)} \\ &= \left(J_{s+r}(X)\right)^{1-1/(2s+2r)}. \end{aligned}$$

Consequently, one has $J_{s+r}(X) \ll 1$, which contradicts the lower bound (2.3) if $X = X_\ell$ is large enough. We may therefore suppose that $T_1 < T_2$, and hence that $J_{s+r}(X) \leqslant 2T_2$.

Given a solution $\mathbf{x}, \mathbf{y}$ of (6.1) counted by T_2, let

$$D(\mathbf{x}, \mathbf{y}) = \prod_{1 \leqslant i < j \leqslant s+r} (x_i - x_j)(y_i - y_j).$$

Also, let $\mathcal{P}$ denote a set of $\lceil (s+r)^2\theta^{-1}\rceil$ prime numbers in $(M, 2M]$. That such a set of primes exists for large enough X is a consequence of the Prime Number Theorem. From the definition of T_2, we have $D(\mathbf{x}, \mathbf{y}) \neq 0$ and

$$|D(\mathbf{x}, \mathbf{y})| < X^{(s+r)^2} \leqslant M^{\mathrm{card}(\mathcal{P})}.$$

We therefore find that for some $p \in \mathcal{P}$ one must have $p \nmid D(\mathbf{x}, \mathbf{y})$. Denote by $T_2(p)$ the number of solutions of (6.1) counted by $J_{s+r}(X)$ in which $x_1, \dots, x_{s+r}$ are distinct modulo p and likewise $y_1, \dots, y_{s+r}$ are distinct modulo p. Then we have shown thus far that

$$J_{s+r}(X) \leqslant 2T_2 \leqslant 2\sum_{p \in \mathcal{P}} T_2(p),$$

whence for some prime number $p \in \mathcal{P}$, one has

$$J_{s+r}(X) \leqslant 2\lceil (s+r)^2\theta^{-1}\rceil T_2(p). \tag{6.2}$$

We next introduce some notation with which to consider more explicitly the residue classes modulo p of a given solution $\mathbf{x}$, $\mathbf{y}$ counted by $T_2(p)$. Let $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ be s-tuples with $1 \leqslant \boldsymbol{\eta}, \boldsymbol{\zeta} \leqslant p$ satisfying the condition that for $1 \leqslant i \leqslant s$, one has $x_i \equiv \eta_i \pmod p$ and $y_i \equiv \zeta_i \pmod p$. Recall the notation introduced prior to the definition (2.13). Then since $x_1, \dots, x_{s+r}$ are distinct modulo p, it follows that $(x_{s+1}, \dots, x_{s+r}) \in \Xi(\boldsymbol{\eta})$, and likewise one finds that $(y_{s+1}, \dots, y_{s+r}) \in$

$\Xi(\boldsymbol{\zeta})$. Then on considering the underlying Diophantine systems, we obtain the relation

$$T_2(p) \leqslant \sum_{1\leqslant \boldsymbol{\eta},\boldsymbol{\zeta}\leqslant p} \oint \Big(\prod_{i=1}^{s} \mathfrak{f}_1(\boldsymbol{\alpha};\eta_i)\mathfrak{f}_1(-\boldsymbol{\alpha};\zeta_i)\Big) \mathfrak{F}(\boldsymbol{\alpha};\boldsymbol{\eta})\mathfrak{F}(-\boldsymbol{\alpha};\boldsymbol{\zeta})\, d\boldsymbol{\alpha}.$$

Write

$$\mathfrak{I}(\boldsymbol{\theta},\psi) = \oint \left|\mathfrak{F}(\boldsymbol{\alpha};\boldsymbol{\theta})^2 \mathfrak{f}_1(\boldsymbol{\alpha};\psi)^{2s}\right| \, d\boldsymbol{\alpha}.$$

Then by applying Hölder's inequality, and again considering the underlying Diophantine systems, we discern that

$$\begin{aligned} T_2(p) &\leqslant \sum_{1\leqslant \boldsymbol{\eta},\boldsymbol{\zeta}\leqslant p} \prod_{i=1}^{s} \left(\mathfrak{I}(\boldsymbol{\eta},\eta_i)\mathfrak{I}(\boldsymbol{\zeta},\zeta_i)\right)^{1/(2s)} \\ &\leqslant \sum_{1\leqslant \boldsymbol{\eta},\boldsymbol{\zeta}\leqslant p} \prod_{i=1}^{s} \left(\mathfrak{I}(\eta_i,\eta_i)\mathfrak{I}(\zeta_i,\zeta_i)\right)^{1/(2s)}. \end{aligned}$$

Hence, on recalling the definition (2.14), we obtain the upper bound

$$\begin{aligned} T_2(p) &\leqslant p^{2s} \max_{1\leqslant \eta\leqslant p} \oint \left|\mathfrak{F}(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_1(\boldsymbol{\alpha};\eta)^{2s}\right| \, d\boldsymbol{\alpha} \\ &= p^{2s} \max_{1\leqslant \eta\leqslant p} \widetilde{I}_1(X;\eta). \end{aligned} \tag{6.3}$$

The mean value $\widetilde{I}_c(X;\eta)$ counts the number of integral solutions of the system (2.9) with

$$1 \leqslant \mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w} \leqslant X, \quad \mathbf{v} \equiv \mathbf{w} \equiv \eta \pmod{p^c},$$

and with

$$[\mathbf{x} \pmod p] \in \Xi(\eta) \quad \text{and} \quad [\mathbf{y} \pmod p] \in \Xi(\eta).$$

Let T_3 denote the number of such solutions in which the s integers $v_1,\dots,v_s$ together occupy at least r distinct residue classes modulo p^{c+1}, and let T_4 denote the corresponding number of solutions in which these integers together lie in at most $r-1$ distinct residue classes modulo p^{c+1}. Then $\widetilde{I}_c(X;\eta) = T_3 + T_4$. By an argument similar to that leading to (4.2), we obtain the bound

$$\begin{aligned} T_3 &\ll \oint |\mathfrak{F}(\boldsymbol{\alpha};\eta)|^2 \, \mathfrak{F}_c(\boldsymbol{\alpha};\eta)\mathfrak{f}_c(\boldsymbol{\alpha};\eta)^{s-r}\mathfrak{f}_c(-\boldsymbol{\alpha};\eta)^s \, d\boldsymbol{\alpha} \\ &\leqslant \left(\oint \left|\mathfrak{F}(\boldsymbol{\alpha};\eta)^2 \mathfrak{F}_c(\boldsymbol{\alpha};\eta)^{2t}\right| \, d\boldsymbol{\alpha}\right)^{1/(2t)} \left(\oint \left|\mathfrak{F}(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_c(\boldsymbol{\alpha};\eta)^{2s}\right| \, d\boldsymbol{\alpha}\right)^{1-1/(2t)} \\ &\leqslant \left(\widetilde{K}_c(X;\eta)\right)^{1/(2t)} \left(\widetilde{I}_c(X;\eta)\right)^{1-1/(2t)}. \end{aligned} \tag{6.4}$$

Also, since $s \geqslant 2r > 2(r-1)$, the argument leading to (4.3) implies that

$$T_4 \ll M \left(\widetilde{I}_c(X;\eta)\right)^{1-3/(2s)} \left(\max_{1\leqslant \zeta\leqslant p^{c+1}} \widetilde{I}_{c+1}(X;\zeta)\right)^{3/(2s)}. \tag{6.5}$$

Then by combining (6.4) and (6.5) to bound $\widetilde{I}_c(X;\eta)$, we infer that

$$\widetilde{I}_c(X;\eta) \ll \widetilde{K}_c(X;\eta) + M^{2s/3} \max_{1\leqslant \zeta \leqslant p^{c+1}} \widetilde{I}_{c+1}(X;\zeta). \tag{6.6}$$

We now iterate (6.6) to bound $\widetilde{I}_1(X;\eta)$, thereby deducing from (6.2), (6.3) and the definition (2.16) that

$$\begin{aligned} J_{s+r}(X) &\ll T_2(p) \\ &\ll \max_{0\leqslant h\leqslant 3} M^{2s}(M^h)^{2s/3} K_{0,1+h}(X) + M^{2s+8s/3} \max_{1\leqslant \zeta\leqslant p^5} \widetilde{I}_5(X;\zeta). \end{aligned} \tag{6.7}$$

By considering the underlying Diophantine systems, we deduce from (2.13) and (2.14) via Corollary 2.2 that

$$\begin{aligned} \widetilde{I}_5(X;\zeta) &\leqslant \oint |\mathfrak{f}_0(\boldsymbol{\alpha};0)^{2r} \mathfrak{f}_5(\boldsymbol{\alpha};\zeta)^{2s}|\, d\boldsymbol{\alpha} \\ &\ll (J_{s+r}(X))^{r/(s+r)} \left(J_{s+r}(X/M^5)\right)^{s/(s+r)}. \end{aligned}$$

Now (6.7) implies either that

$$J_{s+r}(X) \ll M^{2s+2sh/3} K_{0,1+h}(X) \tag{6.8}$$

for some index $h \in \{0,1,2,3\}$, so that the conclusion of the lemma holds, or else that

$$J_{s+r}(X) \ll M^{14s/3} (J_{s+r}(X))^{r/(s+r)} (J_{s+r}(X/M^5))^{s/(s+r)}.$$

In the latter case, since $\lambda \geqslant s+r$, we obtain the upper bound

$$\begin{aligned} J_{s+r}(X) &\ll M^{14(s+r)/3} J_{s+r}(X/M^5) \ll M^{14(s+r)/3} (X/M^5)^{\lambda+\delta} \\ &\ll X^{\lambda+\delta} M^{-(s+r)/3}. \end{aligned}$$

Invoking the definition (2.2) of δ, we find that $J_{s+r}(X) \ll X^{\lambda-2\delta}$, contradicting the lower bound (2.3) if $X = X_\ell$ is large enough. We are therefore forced to accept the former upper bound (6.8), and hence the proof of the lemma is complete. □

7. The iterative process

By first applying Lemma 6.1, and following up with repeated application of Lemma 5.2, we are able to bound $J_{s+r}(X)$ in terms of quantities of the shape $K_{c,d}(X)$, in which c and d pass through an increasing sequence of integral values. In this section we explore this iterative process, and ultimately establish Theorem 1.3.

Lemma 7.1. *Suppose $\Lambda \geqslant 0$. Let a and b be integers with $0 \leqslant a < b \leqslant (32t\theta)^{-1}$ and $b \geqslant ta$, and put $g = b - ta$. Suppose that there are real numbers ψ, c and γ, with*

$$0 \leqslant c \leqslant (2\delta)^{-1}\theta, \quad \gamma \geqslant -rb \quad \text{and} \quad \psi \geqslant 0,$$

such that

$$X^\Lambda M^{\Lambda\psi} \ll X^{c\delta} M^{-\gamma} [[K_{a,b}(X)]]. \tag{7.1}$$

Then, for some integer h with $0 \leqslant h \leqslant 15(t-1)b$, one has

$$X^{\Lambda}M^{\Lambda\psi'} \ll X^{c'\delta}M^{-\gamma'}[[K_{b,tb+h}(X)]],$$

where

$$\psi' = t\psi + (t-1)b, \quad c' = t(c+1), \quad \gamma' = t\gamma + \tfrac{4}{3}sh - sg.$$

Proof. From Lemma 5.2, there exists an integer h with $0 \leqslant h < 15(t-1)b$ with the property that

$$[[K_{a,b}(X)]] \ll X^{\delta}M^{rg}\left(M^{-4sh/3}[[K_{b,tb+h}(X)]]\right)^{1/t}(X/M^b)^{\Lambda(1-1/t)} + (M^{15(t-1)b})^{-r/6}(X/M^b)^{\Lambda}.$$

Consequently, from the hypothesised bound (7.1) we infer that

$$X^{\Lambda}M^{\Lambda\psi} \ll X^{(c+1)\delta}M^{-\gamma+rg}\left(M^{-4sh/3}[[K_{b,tb+h}(X)]]\right)^{1/t}(X/M^b)^{\Lambda(1-1/t)} + X^{c\delta}M^{-\gamma-2rb}X^{\Lambda}.$$

By hypothesis, we have $X^{c\delta} \leqslant M^{1/2}$, whence $X^{c\delta}M^{-\gamma-2rb} \leqslant M^{1/2-rb} \leqslant M^{-1/2}$ and thus

$$X^{\Lambda/t}M^{\Lambda(\psi+(1-1/t)b)} \ll X^{(c+1)\delta}M^{-\gamma+rg-4rh/3}[[K_{b,tb+h}(X)]]^{1/t}.$$

The conclusion of the lemma follows on raising left and right hand sides in the last inequality to the power t. □

Lemma 7.2. *We have $\Lambda \leqslant 0$.*

Proof. Assume that $\Lambda > 0$, for otherwise there is nothing to prove. We begin by noting that as a consequence of Lemma 6.1, it follows from (2.17) and (2.19) that there exists an integer $h_{-1} \in \{0,1,2,3\}$ such that

$$[[J_{s+r}(X)]] \ll (M^{h_{-1}})^{-4s/3}[[K_{0,1+h_{-1}}(X)]].$$

We therefore deduce from (2.20) that

$$X^{\Lambda} \ll X^{\delta}[[J_{s+r}(X)]] \ll X^{\delta}(M^{h_{-1}})^{-4s/3}[[K_{0,1+h_{-1}}(X)]]. \tag{7.2}$$

Next we define sequences (a_n), (b_n), (h_n), (c_n), (γ_n), (ψ_n) for $0 \leqslant n \leqslant N$ in such a way that

$$0 \leqslant h_{n-1} \leqslant 15(t-1)b_{n-1} \quad (n \geqslant 1), \tag{7.3}$$

and

$$X^{\Lambda}M^{\Lambda\psi_n} \ll X^{c_n\delta}M^{-\gamma_n}[[K_{a_n,b_n}(X)]]. \tag{7.4}$$

Given a fixed choice for the sequence (h_n), these sequences are defined by means of the relations

$$a_{n+1} = b_n \quad \text{and} \quad b_{n+1} = tb_n + h_n, \tag{7.5}$$

$$\psi_{n+1} = t\psi_n + (t-1)b_n, \tag{7.6}$$

$$c_{n+1} = t(c_n+1), \tag{7.7}$$

$$\gamma_{n+1} = t\gamma_n + \tfrac{4}{3}sh_n - s(b_n - ta_n). \tag{7.8}$$

We put $a_0 = 0$, $b_0 = 1 + h_{-1}$, $\psi_0 = 0$, $c_0 = 1$ and $\gamma_0 = \frac{4}{3}sh_{-1}$, so that (7.4) holds with $n = 0$ as a consequence of our initial choice of h_{-1} together with

(7.2). We prove by induction that for each integer n with $0 \leqslant n < N$, the sequence $(h_m)_{m=-1}^n$ may be chosen in such a way that

$$0 \leqslant a_n < b_n \leqslant (32t\theta)^{-1}, \quad \psi_n \geqslant 0, \quad \gamma_n \geqslant -rb_n, \quad 0 \leqslant c_n \leqslant (2\delta)^{-1}\theta, \quad (7.9)$$

and so that (7.3) and (7.4) both hold with n replaced by $n+1$.

Suppose that $0 \leqslant n < N$, and suppose also that (7.3) and (7.4) both hold for the index n. We have already shown such to be the case when $n = 0$. We observe first that the relation (7.5) plainly demonstrates that $b_n > a_n$ for all n. Moreover, from (7.3) and (7.5), we see that $b_{n+1} \leqslant 16tb_n$ for all n. By induction, therefore, we deduce that $b_n \leqslant 4(16t)^n$ whence, by invoking (2.2) we find that $b_n \leqslant (32t\theta)^{-1}$ for $0 \leqslant n < N$. It is also apparent from (7.6) and (7.7) that c_n and ψ_n are non-negative for all n. In addition, by iterating (7.7), we have

$$c_n = t^n + t\left(\frac{t^n - 1}{t-1}\right) \leqslant 3t^n \quad (n \geqslant 0). \quad (7.10)$$

Thus, by reference to (2.2) we see that $c_n \leqslant (2\delta)^{-1}\theta$ for $0 \leqslant n < N$.

In order to bound γ_n, we begin by noting from (7.5) that for $m \geqslant 1$,

$$h_m = b_{m+1} - tb_m \quad \text{and} \quad a_m = b_{m-1}.$$

Then it follows from (7.8) that for $m \geqslant 1$ one has

$$\gamma_{m+1} - \tfrac{4}{3}sb_{m+1} + sb_m = t\left(\gamma_m - \tfrac{4}{3}sb_m + sb_{m-1}\right).$$

By iterating this identity, we deduce that for $m \geqslant 1$, one has

$$\gamma_m = \tfrac{4}{3}sb_m - sb_{m-1} + t^{m-1}\left(\gamma_1 - \tfrac{4}{3}sb_1 + sb_0\right).$$

On recalling that $b_0 = 1 + h_{-1}$, $\gamma_0 = \frac{4}{3}sh_{-1}$ and $b_1 = tb_0 + h_0$, we discern first from (7.8) that

$$\gamma_1 = \tfrac{4}{3}st(b_0 - 1) + \tfrac{4}{3}s(b_1 - tb_0) - sb_0 = \tfrac{4}{3}s(b_1 - t) - sb_0,$$

and hence that

$$\gamma_m = \tfrac{4}{3}sb_m - sb_{m-1} - \tfrac{4}{3}st^m \quad (m \geqslant 1). \quad (7.11)$$

Finally, we find from (7.5) that $b_m \geqslant tb_{m-1} \geqslant t^m$ for $m \geqslant 1$, and hence

$$\gamma_m = \tfrac{4}{3}s(b_m - t^m) - sb_{m-1} \geqslant -sb_{m-1} \geqslant -rb_m.$$

Collecting together this conclusion with those of the previous paragraph, we have shown that (7.9) holds for $0 \leqslant n < N$.

At this point in the argument, we may suppose that both (7.4) and (7.9) hold for the index n. An application of Lemma 7.1 therefore reveals that there exists an integer h_n satisfying the constraint implied by (7.3) with n replaced by $n+1$, for which the upper bound (7.4) holds also with n replaced by $n+1$. This completes the inductive step, so that in particular the upper bound (7.4) holds for $0 \leqslant n \leqslant N$.

We now exploit the bound just established. Since we have $b_N \leqslant 4(16t)^N \leqslant (2\theta)^{-1}$, it is a consequence of Lemma 5.3 that

$$[[K_{a_N,b_N}(X)]] \ll X^{\Lambda+\delta}(M^{b_N - b_{N-1}})^s. \quad (7.12)$$

By combining (7.4) with (7.11) and (7.12), we obtain the bound

$$\begin{aligned} X^{\Lambda}M^{\Lambda\psi_N} &\ll X^{\Lambda+(c_N+1)\delta}M^{(b_N-b_{N-1})s-\gamma_N} \\ &= X^{\Lambda+(c_N+1)\delta}M^{(4s/3)t^N-(s/3)b_N}. \end{aligned} \tag{7.13}$$

By applying (7.10) and (2.2), on the other hand, we have

$$X^{(c_N+1)\delta} < M.$$

We therefore deduce from (7.13) and the lower bound $b_N \geqslant t^N$ that

$$\Lambda\psi_N \leqslant \tfrac{4}{3}st^N - \tfrac{1}{3}b_Ns + 1 \leqslant st^N + 1.$$

In addition, a further application of the lower bound $b_n \geqslant t^n$ reveals that

$$\psi_{n+1} = t\psi_n + (t-1)b_n \geqslant t\psi_n + (t-1)t^n,$$

whence $\psi_N \geqslant N(t-1)t^{N-1}$. Thus we deduce that

$$\Lambda \leqslant \frac{st^N+1}{N(t-1)t^{N-1}} \leqslant \frac{3s}{N}.$$

Since N may be taken arbitrarily large in terms of s, we are forced to conclude that $\Lambda \leqslant 0$, and this completes the proof of the lemma. □

The conclusion of Theorem 1.3 is an immediate consequence of Lemma 7.2. For the latter shows that when $s = rt$, then for each $\varepsilon > 0$ one has

$$J_{s+r}(X) \ll X^{2s+2r-\kappa+\varepsilon},$$

where κ is given by (1.7).

8. A mean value estimate for Weyl sums

Our goal in this section is to establish a mean value estimate for one-dimensional Weyl sums that, in a sense, forms a hybrid between the treatments of [2] and [12, §10]. This estimate permits the output from the efficient congruencing method to be more effectively transformed into a mean value estimate for one-dimensional Weyl sums.

Consider natural numbers s and m with $1 \leqslant m \leqslant k$. When $q \in \mathbb{N}$ and $b \in \mathbb{Z}$, we define the quantity $I_{s,m}(X;q,b)$ to be the number of integral solutions of the system of equations

$$\left.\begin{aligned} \sum_{i=1}^{s}\big((qx_i+b)^k - (qy_i+b)^k\big) &= 0, \\ \sum_{i=1}^{s}(x_i^j - y_i^j) &= 0 \quad (1 \leqslant j \leqslant m-1), \end{aligned}\right\} \tag{8.1}$$

with $0 \leqslant \mathbf{x}, \mathbf{y} \leqslant X/q$. We begin by adapting the work of [12, §10] so as to estimate $I_{s,k-1}(X;q,b)$ on average over q. To assist with our discussion, we now define $\eta(s,k)$ to be the least positive number η with the property that, whenever X is sufficiently large in terms of s and k, one has

$$J_{s,k}(X) \ll_{\varepsilon} X^{2s-\frac{1}{2}k(k+1)+\eta+\varepsilon}.$$

Throughout this section and the following section, we adopt the convention that whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement. It is convenient to write

$$f(\boldsymbol{\alpha}; X) = \sum_{1 \leqslant x \leqslant X} \mathrm{e}(\alpha_1 x + \ldots + \alpha_k x^k).$$

We pause to recall a lemma on reciprocal sums.

Lemma 8.1. *Suppose that δ is a positive number, and that α and β are real numbers. Let N and R be large real numbers, and write $B = N^{1+\delta} + R^{1+\delta}$. Then*

$$\sum_{1 \leqslant z \leqslant R} \min\{N, \|z\alpha + \beta\|^{-1}\} \ll B + (\log B) \sum_{1 \leqslant u \leqslant BN^{-\delta}} \min\{NR/u, \|u\alpha\|^{-1}\}.$$

Proof. This is [11, Lemma 3.4]. □

When $\mathcal{Q} \subset \mathbb{N}$, write

$$\Theta_{s,k}(\mathcal{Q}) = \sum_{q \in \mathcal{Q}} \max_{(b,q)=1} I_{s,k-1}(X; q, b).$$

Lemma 8.2. *Let X denote a large positive number, and let Q be a real number with $1 < Q \leqslant X^{(k-2)/(k-1)}$. Suppose that $\mathcal{Q} \subseteq (2^{-k}Q, Q]$ is a set of natural numbers with card$(\mathcal{Q}) \gg Q(\log Q)^{-k}$ satisfying the condition that for each $q \in \mathcal{Q}$, one has $(q, k) = 1$. Then for each natural number s, one has*

$$\Theta_{s,k}(\mathcal{Q}) \ll (X/Q)^{2s - \frac{1}{2}(k^2-k+2)+\varepsilon} \left((X/Q)^{\eta(s,k)-1} + (X/Q)^{\eta(s,k-1)} \right).$$

Proof. For the moment, consider fixed integers q and b with $(kb, q) = 1$ and $2^{-k}Q < q \leqslant Q$. Define $\Upsilon_k(X; h) = \Upsilon_k(X; h; q, b)$ to be the number of integral solutions of the Diophantine system

$$\left.\begin{aligned} \sum_{i=1}^{s} \left((qx_i + b)^k - (qy_i + b)^k \right) &= 0, \\ \sum_{i=1}^{s} (x_i^{k-1} - y_i^{k-1}) &= h, \\ \sum_{i=1}^{s} (x_i^j - y_i^j) &= 0 \quad (1 \leqslant j \leqslant k-2), \end{aligned}\right\} \tag{8.2}$$

with $0 \leqslant \mathbf{x}, \mathbf{y} \leqslant X/q$. Then on considering the corresponding system (8.1), we see that

$$I_{s,k-1}(X; q, b) = \sum_{|h| \leqslant s(X/q)^{k-1}} \Upsilon_k(X; h). \tag{8.3}$$

Next, by applying an integer shift z to the variables in the system (8.2), we find that $\Upsilon_k(X; h)$ counts the number of integral solutions of the Diophantine

system

$$\left.\begin{aligned}\sum_{i=1}^{s}\big((q(x_i-z)+b)^k-(q(y_i-z)+b)^k\big)&=0,\\ \sum_{i=1}^{s}((x_i-z)^{k-1}-(y_i-z)^{k-1})&=h,\\ \sum_{i=1}^{s}((x_i-z)^j-(y_i-z)^j)&=0\quad(1\leqslant j\leqslant k-2),\end{aligned}\right\}$$

with $z\leqslant \mathbf{x},\mathbf{y}\leqslant z+X/q$. By applying the Binomial Theorem, we find that $\mathbf{x},\mathbf{y}$ satisfies this system of equations if and only if

$$\left.\begin{aligned}\sum_{i=1}^{s}(x_i^j-y_i^j)&=0\quad(1\leqslant j\leqslant k-2),\\ \sum_{i=1}^{s}(x_i^{k-1}-y_i^{k-1})&=h,\\ q\sum_{i=1}^{s}\big(x_i^k-y_i^k\big)&=k(qz-b)h.\end{aligned}\right\}\tag{8.4}$$

Notice that, in view of the hypothesis $(kb,q)=1$, the equation of degree k in (8.4) ensures that $q|h$. We write $g=h/q$, so that the condition $|h|\leqslant s(X/q)^{k-1}$ in (8.3) implies that $|g|\leqslant sq^{-1}(X/q)^{k-1}$.

If we restrict the shifts z to lie in the interval $1\leqslant z\leqslant X/q$, then we see that an upper bound for $\Upsilon_k(X;h)$ is given by the number of integral solutions of the system

$$\left.\begin{aligned}\sum_{i=1}^{s}(x_i^j-y_i^j)&=0\quad(1\leqslant j\leqslant k-2),\\ \sum_{i=1}^{s}(x_i^{k-1}-y_i^{k-1})&=qg,\\ \sum_{i=1}^{s}\big(x_i^k-y_i^k\big)&=k(qz-b)g,\end{aligned}\right\}$$

with $1\leqslant \mathbf{x},\mathbf{y}\leqslant 2X/q$. On considering the underlying Diophantine system, we therefore deduce from (8.3) that for each integer z with $1\leqslant z\leqslant X/q$, the mean value $I_{s,k-1}(X;q,b)$ is bounded above by

$$\sum_{|g|\leqslant sq^{-1}(X/q)^{k-1}}\oint|f(\boldsymbol{\alpha};2^{k+1}X/Q)|^{2s}\mathrm{e}(-k(qz-b)g\alpha_k-qg\alpha_{k-1})\,d\boldsymbol{\alpha}.$$

Write

$$\psi_{q,b}(z;\alpha_k,\alpha_{k-1})=\min\{q^{-1}(X/q)^{k-1},\|k(qz-b)\alpha_k+q\alpha_{k-1}\|^{-1}\}$$

and

$$\Psi_{q,b}(\alpha_k,\alpha_{k-1})=\sum_{1\leqslant z\leqslant X/q}\psi_{q,b}(z;\alpha_k,\alpha_{k-1}). \tag{8.5}$$

Then we obtain the estimate

$$\begin{aligned}I_{s,k-1}(X;q,b)&\ll (X/q)^{-1}\sum_{1\leqslant z\leqslant X/q}\oint |f(\boldsymbol{\alpha};2^{k+1}X/Q)|^{2s}\psi_{q,b}(z;\alpha_k,\alpha_{k-1})\,d\boldsymbol{\alpha}\\ &=(X/q)^{-1}\oint |f(\boldsymbol{\alpha};2^{k+1}X/Q)|^{2s}\Psi_{q,b}(\alpha_k,\alpha_{k-1})\,d\boldsymbol{\alpha}. \end{aligned}\tag{8.6}$$

Our assumption that $1<Q\leqslant X^{(k-2)/(k-1)}$ ensures that $X/q\leqslant q^{-1}(X/q)^{k-1}$. Then by applying Lemma 8.1 with $\alpha=kq\alpha_k$, we deduce from (8.5) that

$$\begin{aligned}\Psi_{q,b}(\alpha_k,\alpha_{k-1})\ll{}&q^{-1}(X/q)^{k-1+\varepsilon}\\ &+X^{\varepsilon}\sum_{1\leqslant u\leqslant 2q^{-1}(X/q)^{k-1}}\min\{(qu)^{-1}(X/q)^k,\|kqu\alpha_k\|^{-1}\}.\end{aligned}$$

Define

$$\Phi(\alpha_k,\alpha_{k-1})=\sum_{q\in\mathcal{Q}}\max_{(b,q)=1}\Psi_{q,b}(\alpha_k,\alpha_{k-1}). \tag{8.7}$$

Then we arrive at the upper bound

$$\begin{aligned}\Phi(\alpha_k,\alpha_{k-1})\ll{}&X^{k-1+\varepsilon}\sum_{2^{-k}Q<q\leqslant Q}q^{-k}\\ &+X^{\varepsilon}\sum_{1\leqslant q\leqslant Q}\sum_{1\leqslant u\leqslant 2q^{-1}(X/q)^{k-1}}\min\{(qu)^{-1}(X/Q)^k,\|kqu\alpha_k\|^{-1}\}.\end{aligned}$$

By making use of a familiar estimate for the divisor function, therefore, we obtain the bound

$$\Phi(\alpha_k,\alpha_{k-1})\ll (X/Q)^{k-1+\varepsilon}+X^{\varepsilon}\sum_{1\leqslant v\leqslant k2^{k^2}(X/Q)^{k-1}}\min\{(X/Q)^kv^{-1},\|v\alpha_k\|^{-1}\}.$$

Suppose that $\alpha_k\in\mathbb{R}$, and that $c\in\mathbb{Z}$ and $r\in\mathbb{N}$ satisfy $(c,r)=1$ and $|\alpha_k-c/r|\leqslant r^{-2}$. Then it follows from [6, Lemma 2.2] that

$$\Phi(\alpha_k,\alpha_{k-1})\ll (X/Q)^{k+\varepsilon}\left((X/Q)^{-1}+r^{-1}+r(X/Q)^{-k}\right). \tag{8.8}$$

Applying a standard transference principle (compare Exercise 2 of [6, §2.8]), it follows that

$$\Phi(\alpha_k,\alpha_{k-1})\ll (X/Q)^{k+\varepsilon}\left((X/Q)^{-1}+\mathfrak{H}_{r,c}(\alpha)^{-1}+\mathfrak{H}_{r,c}(\alpha)(X/Q)^{-k}\right), \tag{8.9}$$

where $\mathfrak{H}_{r,c}(\alpha)=r+(X/Q)^k|r\alpha_k-c|$.

We now compare the respective estimates (8.8) and (8.9) on the one hand, and [12, estimates (10.6) and (10.7)] on the other. In this way, one finds that the argument of the proof of [12, Lemma 10.1] leading to the estimate (10.10)

of that paper may be adapted without serious modification to deliver from (8.6) and (8.7) the bound

$$\begin{aligned}\Theta_{s,k}(\mathcal{Q}) &\ll (X/Q)^{-1} \oint |f(\boldsymbol{\alpha}; 2^{k+1}X/Q)|^{2s}\Phi(\alpha_k, \alpha_{k-1})\, d\boldsymbol{\alpha}\\ &\ll (X/Q)^{k-2+\varepsilon} J_{s,k}(2^{k+1}X/Q) + (X/Q)^{\varepsilon-1} J_{s,k-1}(2^{k+1}X/Q)\\ &\ll (X/Q)^{2s-\frac{1}{2}k(k+1)+\varepsilon}\left((X/Q)^{k-2+\eta(s,k)} + (X/Q)^{k-1+\eta(s,k-1)}\right).\end{aligned}$$

The conclusion of the lemma now follows. □

In the next phase of our work in this section, we make use of the iterative process from [2], and this entails the introduction of certain sets of prime numbers. Let X be a large real number and for $r \geqslant 1$ denote by Y_r the set of primes in the interval $(sX^{1/(r(r+1))}, 2sX^{1/(r(r+1))}]$. We adopt the convention in what follows that the empty product is 1.

Lemma 8.3. *Suppose that $k \geqslant 3$, $1 \leqslant m \leqslant k-1$, $s > m$ and $q = p_1 \cdots p_{m-1}$, where each $p_i \in Y_i$. Let $\mathcal{P}_m$ be any set of $2sk^4$ primes in the set Y_m. Also, suppose that b is an integer with $0 \leqslant b < q$ satisfying $(b, q) = 1$. Then*

$$I_{s,m}(X; q, b) \ll \max_{p\in\mathcal{P}_m} p^{2s-2m+\frac{3}{2}m(m+1)} \max_{a\in\mathcal{B}(p)} I_{s-m,m+1}(X; pq, b+aq),$$

where $\mathcal{B}(p) = \mathcal{B}(p; q, b)$ denotes the set of integers a with $0 \leqslant a < p$ and $(b + aq, pq) = 1$.

Proof. This is essentially the special case of [2, Lemma 4.1] in which $f(x) = x^k$. The statement of [2, Lemma 4.1] has the stronger hypotheses that each p_i be one of the *smallest* $2sk^4$ primes in Y_i, and that $\mathcal{P}_m$ be the set of $2sk^4$ smallest primes in Y_m. The argument of the proof, however, shows that the conclusion holds whenever $p_i \in Y_i$ for $1 \leqslant i \leqslant m-1$ and $\mathcal{P}_m \subseteq Y_m$. □

Lemma 8.4. *When $1 \leqslant m \leqslant k-1$, $q \leqslant (2s)^m X^{m/(m+1)}$ and $(b, q) = 1$, one has*

$$I_{s,m}(X; q, b) \ll \Big(\prod_{j=m}^{k-2} q^{-1}(X/q)^j\Big) I_{s,k-1}(X; q, b).$$

Proof. The argument of the proof of [2, Lemma 4.2] shows that for $1 \leqslant m \leqslant k-2$, one has

$$I_{s,m}(X; q, b) \leqslant \left(1 + sq^{-1}(X/q)^m\right) I_{s,m+1}(X; q, b).$$

The desired conclusion therefore follows by induction on m. □

We are now equipped to state and prove the main result of this section. Define the exponential sum $g(\alpha) = g_k(\alpha; X)$ by

$$g_k(\alpha; X) = \sum_{1\leqslant x\leqslant X} \mathrm{e}(\alpha x^k),$$

and when $s \in \mathbb{N}$, define

$$I_s(X) = \int_0^1 |g(\alpha)|^{2s}\, d\alpha.$$

Theorem 8.5. *Let s be a natural number. Then whenever r is a natural number with $1 \leqslant r \leqslant k-1$, one has*

$$I_s(X) \ll X^{2s-k+\varepsilon}\left(X^{\eta_r^*(s,k)-1/r} + X^{\eta_r^*(s,k-1)}\right),$$

where

$$\eta_r^*(s,w) = r^{-1}\eta(s - \tfrac{1}{2}r(r-1), w).$$

Proof. By the Prime Number Theorem, for $1 \leqslant i \leqslant r-1$ there is a collection $\mathcal{C}_i$ of $\lceil X^{1/(i(i+1))}(2sk^4 \log X)^{-1}\rceil$ disjoint sets of $2sk^4$ primes in the set Y_i. Fix some choice of sets $\mathcal{P}_1 \in \mathcal{C}_1$, …, $\mathcal{P}_{r-1} \in \mathcal{C}_{r-1}$. By applying Lemma 8.3, one finds that whenever b and q satisfy the hypotheses of that lemma, then

$$I_{s-\frac{1}{2}m(m-1),m}(X;q,b) \ll X^{\frac{2s}{m(m+1)}+\frac{1}{2}} \max_{p\in\mathcal{P}_m}\max_{a\in\mathcal{B}(p)} I_{s-\frac{1}{2}m(m+1),m+1}(X;pq,b+aq).$$

By iterating this relation, starting with $m=1$ and terminating with Lemma 8.4 at $m=r$, we obtain

$$I_s(X) \ll X^{\Omega} I_{s-\frac{1}{2}r(r-1),k-1}(X;q,b), \tag{8.10}$$

in which

$$\Omega = 2s\sum_{m=1}^{r-1}\frac{1}{m(m+1)} + \frac{r-1}{2} + \sum_{j=r}^{k-2}\left(\frac{j+1}{r}-1\right),$$

and $q = p_1\cdots p_{r-1}$ for some prime numbers $p_i \in \mathcal{P}_i$ $(1 \leqslant i \leqslant r-1)$. A modest computation confirms that

$$\begin{aligned}\Omega &= 2s(1-1/r) + (r-1)/2 + \tfrac{1}{2}k(k-1)/r - \tfrac{1}{2}r(r+1)/r - (k-1-r)\\ &= 2s(1-1/r) + \tfrac{1}{2}k(k-1)/r - k + r.\end{aligned} \tag{8.11}$$

On putting $Q = (2s)^{r-1}X^{1-1/r}$, we see that $2^{-r}Q < q < Q$. Moreover, distinct choices for the $(r-1)$-tuple $\mathcal{P}_1,\dots,\mathcal{P}_{r-1}$ produce distinct numbers q. Therefore, there is a set $\mathcal{Q}$ of integers in the interval $(2^{-r}Q, Q)$ such that (8.10) holds for each $q \in \mathcal{Q}$. We observe that $(q,k)=1$ for every $q\in\mathcal{Q}$, and moreover that

$$\begin{aligned}\operatorname{card}(\mathcal{Q}) &= \prod_{m=1}^{r-1}\operatorname{card}(\mathcal{C}_m) \gg \prod_{m=1}^{r-1}\left(X^{1/(m(m+1))}(\log X)^{-1}\right)\\ &= X^{1-1/r}(\log X)^{1-r} \gg Q(\log Q)^{1-r}.\end{aligned}$$

Since X is large, it follows that we may apply Lemma 8.2 to infer that

$$\begin{aligned}\Theta_{s-\frac{1}{2}r(r-1),k}(\mathcal{Q}) &\ll X^{\varepsilon}(X/Q)^{2s-r(r-1)-\frac{1}{2}(k^2-k+2)}\\ &\quad\times\Big((X/Q)^{\eta(s-\frac{1}{2}r(r-1),k)-1}+(X/Q)^{\eta(s-\frac{1}{2}r(r-1),k-1)}\Big)\\ &\ll X^{\varepsilon}(X^{1/r})^{2s-r(r-1)-\frac{1}{2}(k^2-k+2)}\left(X^{\eta_r^*(s,k)-1/r}+X^{\eta_r^*(s,k-1)}\right).\end{aligned} \tag{8.12}$$

Next, on substituting (8.11) and (8.12) into (8.10), we deduce that

$$\sum_{q\in\mathcal{Q}} I_s(X)\ll X^{2s-k+1-1/r+\varepsilon}\left(X^{\eta_r^*(s,k)-1/r}+X^{\eta_r^*(s,k-1)}\right).$$

But $\mathrm{card}(\mathcal{Q})\gg X^{1-1/r-\varepsilon}$, and so the conclusion of the theorem follows by dividing left and right hand side of the last relation by $\mathrm{card}(\mathcal{Q})$. □

9. Application to Waring's problem

The mean value estimate supplied by our new bounds for $J_{s,k}(X)$ via Theorem 8.5 may be utilised to derive improvements in our understanding of the asymptotic formula in Waring's problem. Before describing our conclusions, we introduce some notation. We define the set of minor arcs $\mathfrak{m}=\mathfrak{m}_k$ to be the set of real numbers $\alpha\in[0,1)$ satisfying the property that, whenever $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$ and $|q\alpha-a|\leqslant(2k)^{-1}X^{1-k}$, then $q>(2k)^{-1}X$. We recall a mean value estimate restricted to minor arcs.

Theorem 9.1. *Suppose that $s\geqslant k^2-1$. Then for each $\varepsilon>0$, one has*

$$\int_{\mathfrak{m}}|g_k(\alpha;X)|^{2s}\,d\alpha\ll X^{2s-k-1+\varepsilon}.$$

Proof. This is [14, Theorem 10.1]. □

For each natural number v, we define

$$\Delta_v^*=\max\{\eta(v,k)-1,\eta(v,k-1)\},$$

where η is defined as in the preamble to Lemma 8.1. Then, for natural numbers v and w we put

$$s_0(k,v,w)=2k^2-2-\frac{2k^2-2-(2v+w^2-w)}{1+\Delta_v^*/w},$$

and then define

$$s_1(k)=\min_{\substack{1\leqslant w\leqslant k-1\\ 2v+w^2-w<2k^2-2}}\min_{v\geqslant 1}s_0(k,v,w).$$

Theorem 9.2. *Suppose that s and k are natural numbers with $k\geqslant 3$ and $s>s_1(k)$. Then there exists a positive number $\delta=\delta(k,s)$ with the property that*

$$\int_{\mathfrak{m}}|g_k(\alpha;X)|^{s}\,d\alpha\ll X^{s-k-\delta}.$$

Proof. The desired conclusion is immediate from Theorem 9.1 in circumstances where $s \geqslant 2k^2-2$, on making use of the trivial estimate $|g_k(\alpha;X)| \leqslant X$. We suppose therefore that $s_1(k) < s < 2k^2-2$. Let v and w be integers with $1 \leqslant w \leqslant k-1$, $v \geqslant 1$ and $2v+w^2-w < 2k^2-2$, for which $s_1(k) = s_0(k,v,w)$. Then by Hölder's inequality, one has

$$\int_{\mathfrak{m}} |g(\alpha)|^s \, d\alpha \leqslant \Big(\int_{\mathfrak{m}} |g(\alpha)|^{2k^2-2} \, d\alpha \Big)^a \Big(\int_0^1 |g(\alpha)|^{2v+w^2-w} \, d\alpha \Big)^{1-a},$$

where

$$a = \frac{s-(2v+w^2-w)}{2k^2-2-(2v+w^2-w)}.$$

By applying Theorem 9.1 and Theorem 8.5 in sequence, one finds that

$$\begin{aligned} \int_{\mathfrak{m}} |g(\alpha)|^s \, d\alpha &\ll X^{\varepsilon} \Big(X^{2k^2-k-3} \Big)^a \Big(X^{2v+w^2-w-k+\Delta_v^*/w} \Big)^{1-a} \\ &= X^{s-k-a+(1-a)\Delta_v^*/w+\varepsilon}. \end{aligned} \tag{9.1}$$

Since we may suppose that

$$s > s_0(k,v,w) = \frac{(2k^2-2)\Delta_v^* + w(2v+w^2-w)}{w+\Delta_v^*},$$

we see that $a > (1-a)\Delta_v^*/w$, and the conclusion of the theorem follows at once from (9.1). □

We now recall some notation associated with the asymptotic formula in Waring's problem. When s and k are natural numbers, let $R_{s,k}(n)$ denote the number of representations of the natural number n as the sum of s kth powers of positive integers. A formal application of the circle method suggests that for $k \geqslant 3$ and $s \geqslant k+1$, one should have

$$R_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{9.2}$$

where

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(q^{-1} \sum_{r=1}^{q} \mathrm{e}(ar^k/q) \right)^s \mathrm{e}(-na/q).$$

Subject to suitable congruence conditions, one has $1 \ll \mathfrak{S}_{s,k}(n) \ll n^{\varepsilon}$, so that the conjectured relation (9.2) represents an honest asymptotic formula. Let $\widetilde{G}(k)$ denote the least integer t with the property that, for all $s \geqslant t$, and all sufficiently large natural numbers n, one has the asymptotic formula (9.2).

The argument following the proof of [13, Lemma 3.1] may be adapted in the present circumstances to show that $\widetilde{G}(k) \leqslant [s_1(k)] + 1$ for $k \geqslant 3$. For each natural number $m \leqslant \frac{1}{2}k$, we find from Theorem 1.2 that when $v = (k-m)^2 + (k-m)$, one has

$$\eta(v,k) - 1 \leqslant m^2 - 1 \quad \text{and} \quad \eta(v,k-1) \leqslant (m-1)^2,$$

so that

$$\Delta_v^* \leqslant m^2 - 1 \quad \text{for} \quad v = (k-m)^2 + (k-m). \tag{9.3}$$

Similarly, again from Theorem 1.2, for each natural number $m \leqslant \frac{1}{2}(k-1)$, we find that when $v = (k-m)^2 - 1$, one has

$$\eta(v,k) - 1 \leqslant m^2 + m - 1 + \frac{m}{k-m-1}$$

and

$$\eta(v,k-1) \leqslant (m-1)^2 + (m-1) + \frac{m-1}{k-m},$$

so that

$$\Delta_v^* \leqslant m^2 + m - 1 + \frac{m}{k-m-1} \quad \text{for} \quad v = (k-m)^2 - 1. \tag{9.4}$$

Employing these exponents (9.3) and (9.4) in order to obtain upper bounds for $s_1(k)$, we obtain the upper bounds for $\widetilde{G}(k)$ recorded in the following corollary.

Corollary 9.3. *One has*

$$\widetilde{G}(12) \leqslant 253, \quad \widetilde{G}(13) \leqslant 299, \quad \widetilde{G}(14) \leqslant 349, \quad \widetilde{G}(15) \leqslant 403, \quad \widetilde{G}(16) \leqslant 460,$$

$$\widetilde{G}(17) \leqslant 521, \quad \widetilde{G}(18) \leqslant 587, \quad \widetilde{G}(19) \leqslant 656, \quad \widetilde{G}(20) \leqslant 729.$$

We note that in each of these bounds, it is (9.4) which is utilised within the formula for $s_1(k)$. One takes $m = 2$ for $k = 12$, and $m = 3$ for $13 \leqslant k \leqslant 20$. Meanwhile, one takes $w = 5$ for $k = 12$, $w = 6$ for $k = 13, 14$, and $w = 7$ for $15 \leqslant k \leqslant 20$.

For comparison, the bounds for $\widetilde{G}(k)$ made available in [14, Corollary 1.7] show that

$$\widetilde{G}(12) \leqslant 255, \quad \widetilde{G}(13) \leqslant 303, \quad \widetilde{G}(14) \leqslant 354, \quad \widetilde{G}(15) \leqslant 410, \quad \widetilde{G}(16) \leqslant 470,$$

$$\widetilde{G}(17) \leqslant 534, \quad \widetilde{G}(18) \leqslant 602, \quad \widetilde{G}(19) \leqslant 674, \quad \widetilde{G}(20) \leqslant 748.$$

For $k \leqslant 11$, the bounds for $\widetilde{G}(k)$ in [14, Corollary 1.7] prove superior to those that follow from the work of this paper. For large values of k, meanwhile, the conclusion of [14, Corollary 1.6] shows that

$$\widetilde{G}(k) \leqslant 2k^2 - k^{4/3} + O(k).$$

We are able to provide a modest improvement in this bound as a consequence of Theorem 9.2.

Corollary 9.4. *When k is a large natural number, one has*

$$\widetilde{G}(k) \leqslant 2k^2 - 2^{2/3}k^{4/3} + O(k).$$

Proof. As we have already noted, one has $\widetilde{G}(k) \leqslant [s_1(k)] + 1$, and so it suffices to bound $s_1(k)$ for large values of k. We take

$$m = [2^{2/3}k^{1/3}], \quad v = (k-m)^2 + (k-m) \quad \text{and} \quad w = [2^{1/3}k^{2/3}],$$

so that from (9.3) one obtains

$$\begin{aligned} s_0(k,v,w) &\leqslant 2k^2 - 2 - \frac{2k^2 - 2 - 2(k^2 - 2mk) - w^2 + O(k)}{1 + m^2/w + O(k^{-2/3})} \\ &= 2k^2 - 2 - \frac{4(2^{2/3}k^{1/3})k - 2^{2/3}k^{4/3} + O(k)}{3 + O(k^{-1/3})} \\ &= 2k^2 - 2^{2/3}k^{4/3} + O(k). \end{aligned}$$

This confirms the conclusion of the corollary. □

References

[1] G. I. Arkhipov and A. A. Karatsuba, *A new estimate of an integral of I. M. Vinogradov*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 751–762 (Russian), Math. USSR-Izv. **13** (1979), 52–62 (English).

[2] K. B. Ford, *New estimates for mean values of Weyl sums*, Internat. Math. Res. Notices (1995), 155–171.

[3] L.-K. Hua, *Additive theory of prime numbers*, American Math. Soc., Providence, RI, 1965.

[4] Yu. V. Linnik, *On Weyl's sums*, Mat. Sbornik (Rec. Math.) **12** (1943), 28–39 (Russian).

[5] O. V. Tyrina, *A new estimate for a trigonometric integral of I. M. Vinogradov*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 363–378 (Russian), Math. USSR-Izv. **30** (1988), 337–351 (English).

[6] R. C. Vaughan, *The Hardy-Littlewood method*, second ed., Cambridge University Press, Cambridge, 1997.

[7] R. C. Vaughan and T. D. Wooley, *A special case of Vinogradov's mean value theorem*, Acta Arith. **79** (1997), 193–204.

[8] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Trav. Inst. Math. Stekloff **23** (1947), 109pp (Russian); English translation by A. A. Davenport and K. F. Roth, Interscience, London (1954).

[9] T. D. Wooley, *Quasi-diagonal behaviour in certain mean value theorems of additive number theory*, J. Amer. Math. Soc. **7** (1994), 221–245.

[10] T. D. Wooley, *A note on simultaneous congruences*, J. Number Theory **58** (1996), 288–297.

[11] T. D. Wooley, *Weyl's inequality and exponential sums over binary forms*, Funct. Approx. Comment. Math. **28** (2000), 83–95.

[12] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing*, Annals of Math. **175** (2012), 1575–1627.

[13] T. D. Wooley, *The asymptotic formula in Waring's problem*, Internat. Math. Res. Notices (2012), no. 7, 1485–1504.

[14] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing, II*, Duke Math. J. **162** (2013), 673–730.

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