A PROBLEM OF RAMANUJAN, ERDŐS AND KÁTAI ON THE ITERATED DIVISOR FUNCTION

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ABSTRACT. We determine asymptotically the maximal order of $\log d(d(n))$, where $d(n)$ is the number of positive divisors of n. This solves a problem first put forth by Ramanujan in 1915.

1 Introduction

Let $d(n)$ denote the number of positive divisors of an integer n. The extreme large values of $d(n)$ were studied by Wigert [10], (see also [4, Theorem 432]). Wigert proved that

$$m_1(x) := \max_{n \leqslant x} \log d(n) \sim (\log 2) \frac{\log x}{\log_2 x}.$$

Here $\log_k x$ denotes the k-th iterate of the logarithm. The lower bound comes from considering integers of the form $N_k = p_1 \cdots p_k$, where p_j denotes the jth smallest prime. Here $d(N_k) = 2^k$, while $\log N_k \sim k \log k$ by the prime number theorem. In his seminal 1915 paper on highly composite numbers [7], Ramanujan gave a more precise asymptotic for $m_1(x)$. At the very end of his paper, Ramanujan posed the problem of finding the extreme large values of $d(d(n))$. By considering integers of the form

$$2^1 \cdot 3^2 \cdot 5^4 \cdots p_k^{p_k - 1}, \tag{1.1}$$

Ramanujan showed that

$$m_2(x) := \max_{n \leqslant x} \log d(d(n)) \geqslant (\sqrt{2} \log 4 + o(1)) \frac{\sqrt{\log x}}{\log_2 x}.$$

The problem of finding the order of $m_2(x)$ has been mentioned in Erdős [1], Ivić [5], and has been mentioned by Ivić in problem sessions in Ottawa [6] and Oberwolfach.

Erdős and Kátai [3] showed $m_2(x) = (\log x)^{1/2} (\log_2 x)^{O(1)}$ (see (4.1) on p. 270 of [3]). Twenty years later Erdős and Ivić [2] improved the upper bound to

$$m_2(x) \ll \left(\frac{\log x \log_2 x}{\log_3 x} \right)^{1/2}.$$

Smati [8, 9] gave a further improvement

$$m_2(x) \ll \sqrt{\log x},$$

the best estimate known to date. Constructions similar to Ramanujan's seem rather natural, and one might expect that $m_2(x) \ll \frac{\sqrt{\log x}}{\log_2 x}$. This is indeed the case, as we now show. More precisely, we prove an asymptotic formula for $m_2(x)$ with an error term.

Theorem 1. *We have*

$$m_2(x) = \frac{\sqrt{\log x}}{\log_2 x} \left(c + O\left(\frac{\log_3 x}{\log_2 x} \right) \right),$$

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where

$$c = \left(8\sum_{j=1}^{\infty}\log^2(1+1/j)\right)^{1/2} = 2.7959802335\dots.$$

In particular, Theorem 1 implies that

$$\limsup_{n\to\infty} \frac{\log d(d(n))\log_2 n}{\sqrt{\log n}} = c.$$

Ramanujan's examples (1.1) are seen to be suboptimal with respect to the constant c, since $\sqrt{2}\log 4 = 1.9605\dots$.

There is a closely related problem, to estimate the extreme values of $\omega(d(n))$, where $\omega(n)$ is the number of distinct prime factors of n. In fact, both Erdős and Ivić [2] and Smati [9] obtained upper bounds for $d(d(n))$ by first bounding $\omega(d(n))$ and then using the elementary inequality $\log d(m) \ll (\log_2 m)\omega(m)$ (see, e.g., Lemme 3.3 of [8] or Lemma 3.2 below). For this problem, Ramanujan's examples (1.1) are essentially optimal, providing the true order and constant in the asymptotic for $w(x) = \max_{n\leqslant x}\omega(d(n))$.

Theorem 2. *We have*

$$w(x) = \frac{\sqrt{\log x}}{\log_2 x}\left(\sqrt{8} + O\left(\frac{\log_3 x}{\log_2 x}\right)\right),$$

Previously, Erdős and Ivić [2] had shown

$$w(x) \ll \left(\frac{\log x \log_3 x}{\log_2 x}\right)^{1/2},$$

and later Smati [8] found the true order $w(x) \ll \frac{\sqrt{\log x}}{\log_2 x}$.

2 The lower bound in Theorem 1

Notation and basic prime number estimates. Throughout, we make use of the asymptotic

$$p_j = j(\log j + \log_2 j + O(1)), \tag{2.1}$$

which is a simple consequence of the prime number theorem with error term $\pi(x) = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$. Here $\pi(x)$ is the number of primes which are $\leqslant x$. We also denote by $\Omega(n)$ the number of prime power divisors of n.

Proof of the lower bound in Theorem 1. Let x be large and define $\varepsilon = 10\frac{\log_3 x}{\log_2 x}$. Let

$$t = \left\lfloor \left(\frac{8\log 2}{c} - \varepsilon\right)\frac{\sqrt{\log x}}{\log_2 x}\right\rfloor, \qquad a_i = \left\lfloor \frac{1}{2^{i/t}-1}\right\rfloor \quad (1\leqslant i\leqslant t), \tag{2.2}$$

and let

$$n = (p_1\cdots p_{a_1})^{p_1-1}(p_{a_1+1}\cdots p_{a_1+a_2})^{p_2-1}\cdots(p_{a_1+\cdots+a_{t-1}+1}\cdots p_{a_1+\cdots+a_t})^{p_t-1}.$$

The Taylor expansion of $\exp(\frac{\log 2}{t})$ shows that $a_1 = \lfloor (2^{1/t}-1)^{-1}\rfloor = t/\log 2 + O(1)$. By (2.2), for every positive integer j, there are $y_j := \lfloor \frac{\log(1+1/j)}{\log 2}t\rfloor$ indices i with $a_i \geqslant j$. Also, $a_1 + \cdots + a_t \ll t\log t$. Using (2.1), we have $\log p_{a_1+\cdots+a_i} \leqslant \log t + 2\log_2 t + O(1)$, hence

$$\log n \leqslant \sum_{i=1}^{t} a_i(p_i-1)\log p_{a_1+\cdots+a_i} \leqslant \left(\log^2 t + 3(\log_2 t)\log t + O(\log t)\right)\sum_{i=1}^{t} i a_i.$$

From $y_j = O(t/j)$ and the definition of c we obtain

$$\begin{aligned}\sum_{i=1}^{t} ia_i = \sum_{j\leqslant a_1} \frac{y_j(y_j+1)}{2} &= \frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{\log(1+1/j)}{\log 2}\right)^2 t^2 + O(t\log t) \\ &= \frac{c^2}{16(\log 2)^2}t^2 + O(t\log t).\end{aligned} \tag{2.3}$$

From the definition of t, $\log t = \frac{1}{2}\log_2 x - \log_3 x + O(1)$ and $\log_2 t = \log_3 x + O(1)$. Thus,

$$\log n \leqslant \left(1 + \frac{2\log_3 x + O(1)}{\log_2 x}\right)\left(1 - \frac{c\varepsilon}{8\log 2}\right)^2 \log x.$$

Hence, if x is large enough, then $n \leqslant x$. From the definition of n above, we have $d(n) = p_1^{a_1}\cdots p_t^{a_t}$. Therefore,

$$\begin{aligned}\log m_2(x) \geqslant \log d(d(n)) &= \sum_{i=1}^{t}\log(a_i+1) = \sum_{j\geqslant 1}(y_j - y_{j+1})\log(j+1) = \sum_{j\geqslant 1} y_j \log(1+1/j) \\ &= \sum_{j\leqslant a_1}\left(\frac{\log^2(1+1/j)}{\log 2}t + O(1/j)\right) \\ &= \frac{c^2}{8\log 2}t + O(\log t) \\ &= \frac{\sqrt{\log x}}{\log_2 x}\left(c + O\left(\frac{\log_3 x}{\log_2 x}\right)\right).\end{aligned} \tag{2.4}$$

□

3 Proof of the upper bound in Theorem 1

Lemma 3.1. *Let $m_N = \min\{m : d(m) = N\}$ and write $m_N = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$. We have*

(i) $\alpha_1 \geqslant \cdots \geqslant \alpha_r$,
(ii) $N'|N$ *implies* $m_{N'} \leqslant m_N$,
(iii) *for each integer* $k \geqslant 1$, *if* $p_j > p_{r+1}^{1/2^k}$, *then* $\Omega(\alpha_j + 1) \leqslant k$.

Remark 1. Using (2.1) and taking $k = 1$, we see from (iii) that if r is large, then $\alpha_j + 1$ is prime for $\sqrt{r} < j \leqslant r$. Also, by (iii), $\Omega(\alpha_j + 1) \ll \log_2 r$ for all j.

Proof. (i) This is trivial and was observed by Ramanujan [7, (32)].

(ii) If $N'|N$, we can find $\alpha_j' \leqslant \alpha_j$ for each j such that $N' = (\alpha_1'+1)\cdots(\alpha_r'+1)$, and clearly $m_{N'} \leqslant p_1^{\alpha_1'}\cdots p_r^{\alpha_r'} \leqslant m_N$.

(iii) If $p_j > p_{r+1}^{1/2^k}$ and $\Omega(\alpha_j+1) > k$, then there are integers a, b with $\alpha_j + 1 = ab$, $a \geqslant 2$ and $b \geqslant 2^k$. Letting

$$m^* = p_j^{b-1}p_{r+1}^{a-1}\prod_{i\neq j} p_i^{\alpha_i},$$

we see that $d(m^*) = d(m_N) = N$, but

$$\frac{m^*}{m_N} = p_j^{b-1-\alpha_j}p_{r+1}^{a-1} = (p_j^{-b}p_{r+1})^{a-1} < 1,$$

a contradiction. □

Lemma 3.2. *For every $\varepsilon > 0$, and for $\omega(n) = s \geqslant 2$ we have*

$$d(n) \ll_\varepsilon \left(\frac{(2+\varepsilon)\log n}{s\log s} \right)^s.$$

Proof. Write the prime factorization of n as $n = q_1^{a_1} \cdots q_s^{a_s}$, where $q_1 < \cdots < q_s$. Using the arithmetic mean - geometric mean inequality and that $q_i \geqslant p_i$, we have

$$d(n) \leqslant \prod_{i=1}^s (2a_i) \leqslant 2^s \prod_{i=1}^s (a_i \log q_i) \prod_{i=1}^s (\log p_i)^{-1} \leqslant \left(\frac{2\log n}{s} \right)^s \frac{(\log s)^{\pi(s)-s}}{\log 2},$$

the last inequality coming from excluding factors corresponding to $3 \leqslant p_i < s$. Finally, the prime number theorem implies $(\log s)^{\pi(s)} \leqslant (\log s)^{O(s/\log s)} \ll_\varepsilon (1+\varepsilon/2)^s$. □

Remark. Lemma 3.2 is fairly sharp. For example, from the inequality $s = \omega(n) \leqslant (1+o(1))\frac{\log n}{\log_2 n}$, and the observation that $m_1(x)$ is monotonically increasing, we immediately obtain Wigert's upper bound for $\log d(n)$.

The following is the key lemma, which explains the constant c.

Lemma 3.3. *Let $a_1, \dots, a_t$ be positive integers.*
(a) we have

$$\sum_{i=1}^t \log(a_i+1) \leqslant \frac{c}{2} \left(\sum_{i=1}^t i a_i \right)^{1/2}.$$

Moreover, the constant $c/2$ is best possible.
(b) If $a_i \geqslant A$ for all i, where A is a positive integer, then

$$\sum_{i=1}^t \log(a_i+1) \leqslant \left(\frac{1+\log^2(A+1)}{A} \sum_{i=1}^t i a_i \right)^{1/2}.$$

Proof. (a) Without loss of generality, suppose $a_1 \geqslant \cdots \geqslant a_t$. Let $y_j = \#\{i : a_i \geqslant j\}$. Then

$$\sum_{i=1}^t i a_i = \sum_{j\geqslant 1} \frac{y_j(y_j+1)}{2} \geqslant \frac{1}{2} \sum_{j\geqslant 1} y_j^2.$$

By partial summation and the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{i=1}^t \log(a_i+1) &= \sum_{j\geqslant 1} (y_j - y_{j+1}) \log(j+1) = \sum_{j\geqslant 1} y_j \log(1+1/j) \\ &\leqslant \left(\sum_{j\geqslant 1} y_j^2 \right)^{1/2} \left(\frac{c^2}{8} \right)^{1/2}. \end{aligned} \tag{3.1}$$

Moreover, the inequality in (3.1) is an equality if and only if for some real Y, $y_j = Y\log(1+1/j)$ for every j. As the y_j are integers, this cannot happen. However, we can come very close to equality in (3.1) by taking t large and choosing the a_i by (2.2), so that $y_j = \lfloor \frac{\log(1+1/j)}{\log 2} t \rfloor$. By (2.3) and (2.4), we have in this case

$$\sum_{i=1}^t \log(a_i+1) = \frac{c^2}{8\log 2} t + O(\log t), \qquad \sum_{i=1}^t i a_i = \frac{c^2}{16(\log 2)^2} t^2 + O(t\log t),$$

whence

$$\sum_{i=1}^t \log(a_i+1) = \frac{c}{2} \left(1 + O\left(\frac{\log t}{t} \right) \right) \left(\sum_{i=1}^t i a_i \right)^{1/2}.$$

(b) Observe that $y_1 = y_2 = \cdots = y_A$. Arguing similarly to (3.1), we obtain

$$\sum_{i=1}^{t} \log(a_i+1) = \frac{\log(A+1)}{A}(y_1+\cdots+y_A) + \sum_{j>A} y_j \log(1+1/j)$$
$$\leqslant \left(\sum_{j\geqslant 1} y_j^2\right)^{1/2} \left(A\left(\frac{\log(A+1)}{A}\right)^2 + \sum_{j>A}\log^2(1+1/j)\right)^{1/2}.$$

Observing that $\log(1+1/j) < 1/j$ and $\sum_{j>A} 1/j^2 < 1/A$, we obtain (b). □

The next lemma is trivial.

Lemma 3.4. *For any positive integer m, $m \geqslant \sum_{p|m} p$.*

Proof of Theorem 1, upper bound. Let n be large, let $N = d(n)$ and factor $N = N'N''$, where

$$N' = u_1^{b_1}\cdots u_w^{b_w}, \qquad N'' = q_1^{a_1}\cdots q_s^{a_s},$$

where $u_1 < \cdots < u_w$, $q_1 < \cdots < q_s$ are primes, $b_i > (\log_2 n)^6$ for every i and $a_i \leqslant (\log_2 n)^6$ for every i.

Write $m_{N'} = p_1^{\beta_1}\cdots p_h^{\beta_h}$. By Lemma 3.1 (ii), $m_{N'} \leqslant m_N \leqslant n$, so that $h \ll \log n$. By Lemma 3.1 (iii), $\Omega(\beta_i+1) \ll \log_2 h \ll \log_3 n$ for every i. Since $d(m_{N'}) = (\beta_1+1)\cdots(\beta_h+1) = N'$, for each $j \leqslant h$ there are $\gg \frac{b_j}{\log_3 n}$ values of i for which $u_j|(\beta_i+1)$. Thus, by Lemma 3.4,

$$\log n \geqslant \log m_{N'} \geqslant (\log 2)\sum_{i=1}^{h}\beta_i \geqslant \frac{\log 2}{2}\sum_{i=1}^{h}(\beta_i+1)$$
$$\geqslant \frac{\log 2}{2}\sum_{i=1}^{h}\sum_{p|(\beta_i+1)} p \gg \sum_{j=1}^{w} u_j \frac{b_j}{\log_3 n} \geqslant \frac{1}{\log_3 n}\sum_{j=1}^{w} j b_j.$$

Combining this estimate with Lemma 3.3 (b) with $A = (\log_2 n)^6$ gives

$$\log d(N') = \sum_{j=1}^{w}\log(b_j+1) \ll \frac{\log_3 n}{(\log_2 n)^3}\left(\sum_{j=1}^{w} j b_j\right)^{1/2} \ll \frac{(\log n)^{1/2}(\log_3 n)^{3/2}}{(\log_2 n)^3}. \tag{3.2}$$

Next, we bound $d(N'')$.

Case 1) If $s \leqslant \frac{(\log n)^{1/2}}{(\log_2 n)^3}$, Lemma 3.2 implies that $\log d(N'') \ll \frac{(\log n)^{1/2}}{(\log_2 n)^2}$.

Case 2) Now suppose that $s > \frac{(\log n)^{1/2}}{(\log_2 n)^3}$. Write $m_{N''} = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$. By Lemma 3.1 (iii),

$$r \leqslant \Omega(N'') = \sum_{j=1}^{s} a_j = \sum_{i=1}^{r}\Omega(\alpha_i+1) \leqslant r + \sum_{k\geqslant 2}\pi(p_{r+1}^{1/2^k}) = r + O((r/\log r)^{1/2}). \tag{3.3}$$

In particular, $r + O((r/\log r)^{1/2}) \geqslant a_1+\cdots+a_s \geqslant s$, so $r \gg s > \frac{(\log n)^{1/2}}{(\log_2 n)^3}$. Thus, for large enough n, $a_1+\cdots+a_s \leqslant r+\sqrt{r}$. Also by Lemma 3.1 (iii), α_j+1 is prime for $j > \sqrt{r}$. Let $\varepsilon = 20\frac{\log_3 n}{\log_2 n}$. By the lower bound on s, and using $a_i \leqslant (\log_2 n)^6$,

$$\sum_{j>s-s^{1-\varepsilon}} a_j \geqslant s^{1-\varepsilon} \geqslant 2\left(s(\log_2 n)^6\right)^{1/2} \geqslant 2\left(\Omega(N'')\right)^{1/2} \geqslant 2\sqrt{r}, \tag{3.4}$$

hence, using (3.3),

$$\sum_{j\leqslant s-s^{1-\varepsilon}} a_j \leqslant \Omega(N'') - 2\sqrt{r} \leqslant r - \sqrt{r}.$$

Using Lemma 3.1 (i), $\alpha_i + 1 = q_1$ for $r - a_1 < i \leqslant r$, and similarly for each $j \leqslant s - s^{1-\varepsilon}$, $\alpha_i + 1 = q_j$ for $r - (a_1 + \cdots + a_j) < i \leqslant r - (a_1 + \cdots + a_{j-1})$. We obtain

$$\begin{aligned}\log m_{N''} &\geqslant \sum_{\sqrt{r}<i\leqslant r} \alpha_i \log p_i \geqslant \sum_{j\leqslant s-s^{1-\varepsilon}} (q_j - 1) \sum_{m=r-(a_1+\cdots+a_j)+1}^{r-(a_1+\cdots+a_{j-1})} \log p_m \\ &\geqslant \sum_{j\leqslant s-s^{1-\varepsilon}} (p_j - 1) a_j \log(r - (a_1 + \cdots + a_j)).\end{aligned}$$

By (3.4), uniformly for $j \leqslant s - s^{1-\varepsilon}$ we have

$$r - (a_1 + \cdots + a_j) = r - \Omega(N'') + a_{j+1} + \cdots + a_s \geqslant s - j - \sqrt{r} \geqslant \frac{1}{2} s^{1-\varepsilon}.$$

Using (2.1), $p_j \geqslant j \log j + 1$ for large j. Hence, by Lemma 3.1 (ii),

$$\begin{aligned}\log n \geqslant \log m_{N''} &\geqslant \sum_{s^{1-\varepsilon}\leqslant j\leqslant s-s^{1-\varepsilon}} (j \log j) a_j (\log s + O(\log_3 n)) \\ &\geqslant (1 + O(\varepsilon)) \frac{(\log_2 n)^2}{4} \sum_{s^{1-\varepsilon}\leqslant j\leqslant s-s^{1-\varepsilon}} j a_j.\end{aligned}$$

By the definition of ε, $s^\varepsilon \gg (\log_2 n)^9$. Also, trivially $\sum_{j=1}^s j a_j \geqslant 1 + 2 + \cdots + s \geqslant \frac{1}{2} s^2$. Recalling that $a_j \leqslant (\log_2 n)^6$ for every j, we have

$$\begin{aligned}\sum_{s^{1-\varepsilon}\leqslant j\leqslant s-s^{1-\varepsilon}} j a_j &= \sum_{j=1}^s j a_j + O\left(s^{2-\varepsilon} (\log_2 n)^6\right) = \sum_{j=1}^s j a_j + O(s^2 (\log_2 n)^{-3}) \\ &= (1 + O(1/\log_2 n)) \sum_{j=1}^s j a_j.\end{aligned}$$

Combining the last two inequalities gives

$$\log n \geqslant \left(1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right) \frac{(\log_2 n)^2}{4} \sum_{j=1}^s j a_j.$$

Applying Lemma 3.3 (a), we conclude that

$$\log d(N'') = \sum_{j=1}^s \log(a_j + 1) \leqslant \frac{c}{2} \Big(\sum_{j=1}^s j a_j\Big)^{1/2} \leqslant c \frac{\sqrt{\log n}}{\log_2 n} \left(1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right). \tag{3.5}$$

Recall that we have a smaller upper bound for $\log d(N'')$ in case 1). Finally, using $d(d(n)) = d(N')d(N'')$ and combining (3.2) and (3.5), we obtain the desired upper bound for $d(d(n))$. □

4 Proof of Theorem 2

Proof of Theorem 2. For the lower bound, let x be large and put $n = \prod_{i=1}^s p_i^{p_i-1}$, where s is the largest integer such that $n \leqslant x$. Recall that p_j is the j-th smallest prime. Then $d(n) = \prod_{i=1}^s p_i$, thus $\omega(d(n)) = s$. By (2.1),

$$\log n = \sum_{i=1}^s (p_i - 1) \log p_i = \sum_{i=1}^s i \log^2 i + O(i \log i \log_2 i) = \frac{1}{2} s^2 \log^2 s + O(s^2 \log s \log_2 s).$$

Solving for s gives $s = \frac{\sqrt{8\log n}}{\log_2 n} + O(\frac{\sqrt{\log n}\log_3 n}{\log_2^2 n})$. We now prove a lower bound on n. Since $p_{s+1} \sim s\log s \sim \sqrt{2\log n} \ll \sqrt{\log x}$ by (2.1), we have

$$x \geqslant n \geqslant x p_{s+1}^{-p_{s+1}} = x \exp\left(-O\left(\sqrt{\log x}\log_2 x\right)\right).$$

That is, $\log n = \log x + O(\sqrt{\log x}\log_2 x)$. Therefore, $s = \frac{\sqrt{8\log x}}{\log_2 x} + O(\frac{\sqrt{\log x}\log_3 x}{\log_2^2 x})$.

Now let n be a large, positive integer factored as $n = n_1 n_2$, $n_1 = \prod_{i=1}^{r} q_i^{a_i}$, $n_2 = \prod_{i=1}^{r'} (q_i')^{a_i'}$, where q_i, q_i' are primes, $q_i > P$ and $q_i' \leqslant P$ for each i, where $P = \frac{\sqrt{\log n}}{\log_2 n}$. We have

$$\omega(d(n)) \leqslant \omega(d(n_1)) + \omega(d(n_2)). \tag{4.1}$$

Since $\omega(n_2) \leqslant \pi(P) \ll \frac{\sqrt{\log n}}{(\log_2 n)^2}$, Lemma 3.2 implies $\log d(n_2) \ll \sqrt{\log n}/\log_2 n$. Applying the elementary inequality $\omega(u) \ll \frac{\log u}{\log_2 u}$ gives

$$\omega(d(n_2)) \ll \frac{\sqrt{\log n}}{(\log_2 n)^2}. \tag{4.2}$$

Next,

$$\log n_1 \geqslant (\log P)\sum_{i=1}^{r} a_i = \left(\frac{\log_2 n}{2} - \log_3 n\right)\sum_{i=1}^{r} a_i.$$

Letting $s = \omega(d(n_1)) = \omega(\prod(a_i+1))$, Lemma 3.4 implies that

$$\sum_{i=1}^{r} a_i \geqslant \sum_{i=1}^{r}\sum_{p|(a_i+1)}(p-1) \geqslant \sum_{i=1}^{s}(p_i - 1) \geqslant \sum_{i=1}^{s}(i\log i + O(1)) = \frac{1}{2}s^2\log s + O(s^2).$$

Here we used the one-sided inequality $p_i \geqslant i\log i + O(1)$ deduced from (2.1). Thus,

$$\log n \geqslant \log n_1 \geqslant \left(\frac{1}{4} + O\left(\frac{\log_3 n}{\log_2 n}\right)\right)(\log_2 n)s^2\log s + O(s^2\log_2 n).$$

Consider two cases: (i) $s \leqslant \frac{\sqrt{\log n}}{\log_2 n}$, (ii) $s > \frac{\sqrt{\log n}}{\log_2 n}$. In case (ii), we have $\frac{\log n}{\log_2^2 n} \geqslant (\frac{1}{8} + O(\frac{\log_3 n}{\log_2 n}))s^2$, and we obtain in both cases

$$\omega(d(n_1)) = s \leqslant \frac{\sqrt{8\log n}}{\log_2 n} + O\left(\frac{\sqrt{\log n}\log_3 n}{\log_2^2 n}\right),$$

Combining this inequality with (4.1) and (4.2), we obtain the desired upper bound for $\omega(d(n))$. □

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