THE PRIME NUMBER RACE AND ZEROS OF L-FUNCTIONS OFF THE CRITICAL LINE

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Abstract. We examine the effects of certain hypothetical configurations of zeros of Dirichlet L-functions lying off the critical line on the distribution of primes in arithmetic progressions.

1. Introduction

Let $\pi_{q,a}(x)$ denote the number of primes $p \leqslant x$ with $p \equiv a \pmod q$. The study of the relative magnitudes of the functions $\pi_{q,a}(x)$ for a fixed q and varying a is known colloquially as the "prime race problem" or "Shanks-Rényi prime race problem". Fix q and distinct residues $a_1, \dots, a_r$ with $(a_i, q) = 1$ for each i. As colorfully described in the first paper of [KT1], consider a game with r players called "1" through "r", and at time t, each player "j" has a score of $\pi_{q,a_j}(t)$ (i.e. player "j" scores 1 point whenever t reaches a prime $\equiv a_j \pmod q$). As $t \to \infty$, will each player take the lead infinitely often? More generally, will all $r!$ orderings of the players occur for infinitely many integers t? It is generally believed that the answers to both questions is yes, for all $q, a_1, \dots, a_r$.

As first noted by Chebyshev [Ch] in 1853, some orderings may occur far less frequently than others (e.g. if $q = 3$, $a_1 = 1$, $a_2 = 2$, then player "1" takes the lead for the first time when $t = 608,981,813,029$ [BH]). More generally, when $r = 2$, a_1 is a quadratic residue modulo q, and a_2 is a quadratic non-residue modulo q, $\pi_{q,a_2}(x) - \pi_{q,a_1}(x)$ tends to be positive more often than it is negative (this phenomenon is now called "Chebyshev's bias"). In 1914, Littlewood [L] proved that both functions $\pi_{4,3}(x) - \pi_{4,1}(x)$ and $\pi_{3,2}(x) - \pi_{3,1}(x)$ change sign infinitely often. Later Knapowski and Turàn ([KT1], [KT2]) proved for many q, a, b that $\pi_{q,b}(x) - \pi_{q,a}(x)$ changes sign infinitely often. The distribution of the functions $\pi_{q,a}(x)$ is closely linked with the distribution of the zeros in the critical strip $0 < \Re s < 1$ of the Dirichlet L-functions $L(s,\chi)$ for the characters χ modulo q. Some of the results of Knapowski and Turàn are proved under the assumption that the

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functions $L(s,\chi)$ have no real zeros in $(0,1)$, or that for some number K_q, the zeros of the functions $L(s,\chi)$ with $|\Im s| \leqslant K_q$ all have real part equal to $\frac{1}{2}$.

Theoretical results for $r > 2$ are more scant, all depending on the unproven Extended Riemann Hypothesis for q (abbreviated ERH_q), which states that all these zeros lie on the critical line $\Re s = \frac{1}{2}$. Kaczorowski ([K1], [K2], [K3]) has shown that the truth of ERH_q implies that for many r-tuples $(q, a_1, \cdots, a_r)$, $\pi_{q,a_1}(x) > \cdots > \pi_{q,a_r}(x)$ for arbitrarily large x. If, in addition to ERH_q, one assumes that the collection of non-trivial zeros of the L-functions for characters modulo q are linearly independent over the rationals (GSH_q, the grand simplicity hypothesis) , Rubinstein and Sarnak [RS] have shown that for any r-tuple of coprime residue classes $a_1, \ldots, a_r$ modulo q, that all $r!$ orderings of the functions $\pi_{q,a_i}(x)$ occur for infinitely many integers x. In fact they prove more, that the logarithmic density of the set of real x for which any such inequality occurs exists and is positive.

In light of the results of Littlewood and of Knapowski and Turàn, one may ask if such results for $r > 2$ may be proved without the assumption of ERH_q. In particular, can it be shown, for some quadruples (q, a_1, a_2, a_3), that the 6 orderings of the functions $\pi_{q,a_i}(x)$ occur for infinitely many integers x, without the assumption of ERH_q (while still allowing the assumption that zeros with imaginary part $< K_q$ lie on the critical line for some constant K_q)? In this paper we answer this question in the negative (in a sense) for all quadruples (q, a_1, a_2, a_3). Thus, in a sense the hypothesis ERH_q is a necessary condition for proving any such results when $r > 2$.

Let C_q be the set of non-principal characters modulo q. Let $D = (q, a_1, a_2, a_3)$, where a_1, a_2, a_3 are distinct residues modulo q which are coprime to q. Suppose for each $\chi \in C_q$, $B(\chi)$ is a sequence of complex numbers with positive imaginary part (possibly empty, duplicates allowed), and denote by $\mathcal{B}$ the system of $B(\chi)$ for $\chi \in C_q$. Let $n(\rho,\chi)$ be the number of occurrences of the number ρ in $B(\chi)$. The system $\mathcal{B}$ is called a *barrier* for D if the following hold:

(i) all numbers in each $B(\chi)$ have real part in $[\beta_2, \beta_3]$, where $\frac{1}{2} < \beta_2 < \beta_3 \leqslant 1$;

(ii) for some β_1 satisfying $\frac{1}{2} \leqslant \beta_1 < \beta_2$, if we assume that for each $\chi \in C_q$ and $\rho \in B(\chi)$, $L(s,\chi)$ has a zero of multiplicity $n(\rho,\chi)$ at $s = \rho$, and all other zeros of $L(s,\chi)$ in the upper half plane have real part $\leqslant \beta_1$, then one of the six orderings of the three functions $\pi_{q,a_i}(x)$ does not occur for large x.

If each sequence $B(\chi)$ is finite, we call $\mathcal{B}$ a *finite barrier* for D and denote by $|\mathcal{B}|$ the sum of the number of elements of each sequence $B(\chi)$, counted according to multiplicity.

Theorem 1. *For every real numbers $\tau > 0$ and $\sigma > \frac{1}{2}$ and every $D = (q, a_1, a_2, a_3)$, there is a finite barrier for D, where each sequence $B(\chi)$ consists of numbers with real part $\leqslant \sigma$ and imaginary part $> \tau$. In fact, for most D, there is a barrier with $|\mathcal{B}| \leqslant 3$.*

We do not claim that the falsity of ERH_q implies that one of the six orderings does not occur for large x. For example, take $\sigma > \frac{1}{2}$, and suppose each non-principal character modulo q has a unique zero with positive imaginary part to the right of the critical line, at $\sigma + i\gamma_\chi$. If the numbers γ_χ are linearly independent over the rationals, it follows easily from Lemma 1.1 below and the Kronecker-Weyl Theorem that in fact all $\phi(q)!$ orderings of the functions $\{\pi_{q,a}(x) : (a,q) = 1\}$ occur for an unbounded set of x.

We now present a general formula for $\pi_{q,a}(x)$ in terms of the zeros of the functions $L(s,\chi)$. Throughout this paper, constants implied by the Landau $O-$ and Vinogradov $\ll -$ symbols may depend on q, but not on any other variable.

Lemma 1.1. *Let $\beta \geqslant \frac{1}{2}$, $x \geqslant 10$ and for each $\chi \in C_q$, let $B(\chi)$ be the sequence of zeros (duplicates allowed) of $L(s,\chi)$ with $\Re s > \beta$ and $\Im s > 0$. Suppose further that all $L(s,\chi)$ are zero-free on the real segment $0 < s < 1$. If $(a,q) = (b,q) = 1$ and x is sufficiently large, then*

$$\phi(q)\left(\pi_{q,a}(x) - \pi_{q,b}(x)\right) = -2\Re\left[\sum_{\chi\in C_q}(\overline{\chi}(a) - \overline{\chi}(b)) \sum_{\substack{\rho\in B(\chi)\\ |\Im\rho|\leqslant x}} f(\rho)\right] + O(x^{\beta}\log^2 x),$$

where

$$f(\rho) := \frac{x^{\rho}}{\rho\log x} + \frac{1}{\rho}\int_2^x \frac{t^{\rho}}{t\log^2 t}\,dt = \frac{x^{\rho}}{\rho\log x} + O\left(\frac{x^{\Re\rho}}{|\rho|^2\log^2 x}\right).$$

Proof. Let $\Lambda(n)$ be the von Mangolt function, and define

$$\Psi_{q,a}(x) = \sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}} \Lambda(n), \qquad \Psi(x;\chi) = \sum_{n\leqslant x}\Lambda(n)\chi(n).$$

Let D_q be the set of all Dirichlet characters χ modulo q. Then

$$\begin{aligned}
\pi_{q,a}(x) &= \sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}} \frac{\Lambda(n)}{\log n} + O(x^{1/2})\\
&= \int_{2-}^{x} \frac{d\Psi_{q,a}(t)}{\log t} + O(x^{1/2})\\
&= \frac{\Psi_{q,a}(x)}{\log x} + \int_2^x \frac{\Psi_{q,a}(t)}{t\log^2 t}\,dt + O(x^{1/2})\\
&= \frac{1}{\phi(q)}\sum_{\chi\in D_q}\overline{\chi}(a)\left(\frac{\Psi(x;\chi)}{\log x} + \int_2^x \frac{\Psi(t;\chi)}{t\log^2 t}\,dt\right) + O(x^{1/2}).
\end{aligned}$$

Then
(1.1)

$$\phi(q)(\pi_{q,a}(x) - \pi_{q,b}(x)) = \sum_{\chi\in C_q}(\overline{\chi}(a) - \overline{\chi}(b))\left(\frac{\Psi(x;\chi)}{\log x} + \int_2^x \frac{\Psi(t;\chi)}{t\log^2 t}\,dt\right) + O(x^{1/2}).$$

By well-know explicit formulas (Ch. 19, (7)–(8) in [D]), when $\chi \in C_q$,

$$\Psi(x;\chi) = -\sum_{|\Im\rho|\leqslant x} \frac{x^\rho}{\rho} + O\left(\log^2 x\right), \tag{1.2}$$

where the sum is over zeros ρ of $L(s,\chi)$ with $0 < \Re\rho < 1$. Since the number of zeros with $0 \leqslant \Im\rho \leqslant T$ is $O(T\log T)$ ([D], Ch. 16, (1)), by partial summation we have

$$\sum_{\substack{0<\Im\rho\leqslant x\\ \Re\rho\leqslant\beta}} \left|\frac{x^\rho}{\rho}\right| \leqslant x^\beta \sum_{0<\Im\rho\leqslant x} \frac{1}{|\rho|} \ll x^\beta \log^2 x.$$

The implied constant depends on the character, and hence only on q. By (1.2),

$$\Psi(x;\chi) = -\sum_{\substack{|\Im\rho|\leqslant x\\ \Re\rho>\beta}} \frac{x^\rho}{\rho} + O\left(x^\beta\log^2 x\right), \tag{1.3}$$

The first part of the lemma follows by inserting (1.3) into (1.1) and combining zeros ρ of $L(s,\chi)$ and $\overline{\rho}$ of $L(s,\overline{\chi})$. Lastly, if $\frac12 \leqslant \sigma = \Re\rho$, integration by parts gives

$$\begin{aligned}\int_2^x \frac{t^\rho}{t\log^2 t}\,dt &= \left.\frac{t^\rho}{\rho\log^2 t}\right|_2^x + \frac{2}{\rho}\int_2^x \frac{t^{\rho-1}}{\log^3 t}\,dt \\ &\ll \frac{x^\sigma}{|\rho|\log^2 x} + \frac{1}{|\rho|}\left[\frac{1}{\log^3 2}\int_2^{\sqrt{x}} t^{\sigma-1}\,dt + \frac{8}{\log^3 x}\int_{\sqrt{x}}^x t^{\sigma-1}\,dt\right] \\ &\ll \frac{x^\sigma}{|\rho|\log^2 x}.\end{aligned}$$

This completes the proof of the lemma. □

In the next three sections, we show several methods for constructing barriers, which, by Lemma 1.1, boils down to analyzing the two functions

$$\Re\sum_{\chi\in C_q} (\overline{\chi}(a_j) - \overline{\chi}(a_3)) \sum_{\rho\in B(\chi)} \frac{x^\rho}{\rho} \quad (j=1,2).$$

In section 2 we construct a barrier using two simple zeros (one of which may be a zero for several characters). Section 3 details a method using a zero for $L(s,\chi)$ and a zero for $L(s,\chi^2)$ (for most D these are simple or double zeros). Lastly, section 4 presents a more general method with two numbers, which are zeros for each character of certain high multiplicities. Together, the three constructions provide barriers for all quadruples (q,a_1,a_2,a_3).

All of the constructions in sections 2–4 involve two zeros, one with imaginary part t and the other with imaginary part $2t$. Thus, we assume that both ERH_q and GSH_q are false. Answering a question posed by Peter Sarnak, in section 5 we construct a barrier (with an infinite set $B(\chi)$) where the imaginary parts of the numbers in the sets $B(\chi)$ are linearly independent; in particular, we assume all zeros of each $L(s,\chi)$ are simple, and $L(s,\chi_1)=0=L(s,\chi_2)$ does not occur for $\chi_1\neq\chi_2$ and $\Re s>\beta_2$.

We adopt the notations $e(z)=e^{2\pi iz}$, $\lfloor x\rfloor$ is the greatest integer $\leqslant x$, $\lceil x\rceil$ is the least integer $\geqslant x$, $\{x\}=x-\lfloor x\rfloor$ is the fractional part of x, and $\|x\|$ is the distance from x to the nearest integer. Also, $\arg z$ is the argument of the nonzero complex number z lying in $[-\pi,\pi)$. Throughout, $q=5$ or $q\geqslant 7$, and $(a_1,q)=(a_2,q)=(a_3,q)=1$.

2. First construction

Lemma 2.1. *If, for some relabelling of the numbers a_i, there is a set S of non-principal Dirichlet characters modulo q such that*

$$\sum_{\chi\in S}\chi(a_1)=\sum_{\chi\in S}\chi(a_2)\neq\sum_{\chi\in S}\chi(a_3),$$

then there is a barrier $\mathcal{B}$ for $D=(q,a_1,a_2,a_3)$ with $|\mathcal{B}|\leqslant|S|+1$.

Remark. The hypotheses of Lemma 2.1 are satisfied when, for example, q has a primitive root g, and a_3/a_2 is not in the subgroup of $(\mathbb{Z}/q\mathbb{Z})^*$ generated by a_2/a_1. Writing $a_2/a_1\equiv g^f$, we take the character with $\chi(g)=e(1/(f,\phi(q)))$ and $S=\{\chi\}$.

Proof. Suppose $1/2\leqslant\beta<\sigma_2<\sigma_1\leqslant\min(\sigma,0.501)$, and let χ_2 be a character with $\chi_2(a_1)\neq\chi_2(a_2)$ (χ_2 may or may not be in S). Let T_q be a large number, depending only on q. Let $\rho_1=\sigma_1+it$, $\rho_2=\sigma_2+2it$ where $t>T_q$. Suppose $L(s,\chi)$ has a simple zero at $s=\rho_1$ for each $\chi\in S$, $L(s,\chi_2)$ has a simple zero at $s=\rho_2$, and no other non-trivial zeros of any L-function in C_q have real part exceeding β. Let

$$D_1(x):=\phi(q)(\pi_{q,a_1}(x)-\pi_{q,a_2}(x)),\qquad D_2(x):=\phi(q)(\pi_{q,a_3}(x)-\pi_{q,a_2}(x)).$$

By Lemma 1.1 and our hypotheses, if x is sufficiently large,

$$D_1(x)=\frac{2x^{\sigma_2}}{\log x}\left[\Re\left(\frac{e^{2it\log x}}{\sigma_2+2it}W\right)+O\left(\frac{1}{\log x}\right)\right],\quad W=\overline{\chi}_2(a_2)-\overline{\chi}_2(a_1),$$
$$D_2(x)=\frac{2x^{\sigma_1}}{\log x}\left[\Re\left(\frac{e^{it\log x}}{\sigma_1+it}Z\right)+O\left(\frac{1}{\log x}\right)\right],\quad Z=\sum_{\chi\in S}\left(\overline{\chi}(a_2)-\overline{\chi}(a_3)\right).$$

Define

$$A(x)=\left\|\frac{1}{\pi}\arg\left(\frac{e^{it\log x}}{\sigma_1+it}Z\right)-\frac{1}{2}\right\|$$
$$=\left\|\frac{1}{\pi}(t\log x+\arg Z+\tan^{-1}(\sigma_1/t))\right\|.$$

If $A(x) \geqslant (\log x)^{-1/2}$, then $|D_2(x)| \gg x^{\sigma_1}/\log^{3/2} x$. But $D_1(x) = O(x^{\sigma_2})$, so for such x, $\pi_{q,a_3}(x)$ is either the largest or the smallest of the three functions. When $A(x) < (\log x)^{-1/2}$, then

$$\begin{aligned} C(x) &:= \arg\left(\frac{e^{2it\log x}}{\sigma_2 + 2it} W\right) \\ &\equiv \arg W - \frac{\pi}{2} + \tan^{-1}\left(\frac{\sigma_2}{2t}\right) + 2t\log x \\ &\equiv \arg W + \tan^{-1}\left(\frac{\sigma_2}{2t}\right) - 2\arg Z - 2\tan^{-1}\left(\frac{\sigma_1}{t}\right) + O\left(\frac{1}{\sqrt{\log x}}\right) \\ &\equiv \arg W - 2\arg Z - F(x) \pmod{\pi}, \end{aligned}$$

where $1/(2t) < F(x) < 1/t$ for large x. The number of possibilities for $\arg W - 2\arg Z$ depends only on q, hence we may assume either

$$B = \left\{\frac{1}{\pi}(\arg W - 2\arg Z)\right\} - \frac{1}{2}$$

satisfies either $B = 0$ or $|B| > 2/t \geqslant 2F(x)$ (by taking T_q sufficiently large). We have

$$C(x) \equiv \pi B + \frac{\pi}{2} - F(x) \pmod{\pi}.$$

If $B = 0$, then $C(x)$ is either $\pi/2 - F(x)$ or $3\pi/2 - F(x)$ (mod 2π), whence $D_1(x)$ takes only one sign for such x. Likewise, $C(x) \in (\pi/2 + 2/t, \pi)$ if $B > 2/t$ and $C(x) \in (-F(x), \pi/2 - 2/t)$ if $B < -2/t$. In all cases, when $A(x) < (\log x)^{-1/2}$, $D_1(x)$ takes only one sign. Therefore, one of the orderings $\pi_{q,a_1}(x) > \pi_{q,a_3}(x) > \pi_{q,a_2}(x)$ or $\pi_{q,a_2}(x) > \pi_{q,a_3}(x) > \pi_{q,a_1}(x)$ does not occur for large x. □

Remark. By similar reasoning, for any integer $k \geqslant 2$ one may construct a barrier with one zero having imaginary part t and another zero having imaginary part kt.

3. Second construction

The basic idea of this section is to find a character χ so that the values $\chi(a_1)$, $\chi(a_2)$, $\chi(a_3)$ are nicely spaced around the unit circle, but not too well spaced (e.g. cube roots of 1 or translates thereof). In almost all circumstances we can find such a character.

Lemma 3.1. *Let $s_1 = ord_q(a_2/a_1)$, $s_2 = ord_q(a_3/a_2)$ and $s_3 = ord_q(a_1/a_3)$. If one of s_1, s_2, s_3 is not in $\{3, 7, 13, 21\}$, then for some relabeling of the a_i's, there is a Dirichlet character χ satisfying either*

(i) *$\chi(a_1) = \chi(a_2) \neq \chi(a_3)$; or*
(ii) *$\chi(a_i) = e(r_i)$ with $0 \leqslant r_1 < r_2 < r_3 < 1$, and $d_1 = r_2 - r_1$, $d_2 = r_3 - r_2$ satisfy*

$$\frac{1}{3} < d_1 \leqslant d_2 < \frac{1}{2}, \quad \text{or} \quad (d_1, d_2) \in \left\{\left(\tfrac{6}{19}, \tfrac{9}{19}\right), \left(\tfrac{12}{37}, \tfrac{16}{37}\right)\right\}. \tag{3.1}$$

Remark. In the case that (i) holds, the hypotheses of Lemma 2.1 hold with $S = \{\chi\}$, and thus there is a finite barrier for D with $|\mathcal{B}| = 2$. Therefore, in this section we confine ourselves with the case that (ii) holds (Lemma 3.5 below).

Before proving Lemma 3.1, we begin with some simple lemmas about the existence of characters with certain properties.

Lemma 3.2. *Suppose $q \geqslant 3$ and $(b,q) = 1$. Let m be the order of b modulo q. Then there is a Dirichlet character χ modulo q with $\chi(b) = e(1/m)$.*

Proof. Suppose $g_1, \dots, g_t$ generate $(\mathbb{Z}/q\mathbb{Z})^*$ and $b = g_1^{f_1} \cdots g_t^{f_t}$. Let $s_i = \text{ord}_q g_i$ for each i, and s_i' be the order of $g_i^{f_i}$. Then $s_i' = s_i/(f_i, s_i)$ and $m = \text{lcm}[s_1', \dots, s_t']$. Let $f_i' = f_i/(f_i, s_i)$, so in particular $(s_i', f_i') = 1$. The gcd of the $t+1$ numbers $m, f_i' m/s_i'$ is 1, so there are integers $h_1, \dots, h_t$ so that $\sum h_i \frac{f_i' m}{s_i'} \equiv 1 \pmod{m}$. Take the character χ with $\chi(g_i) = e(h_i/s_i)$ for each i, then $\chi(b) = \prod \chi(g_i)^{f_i} = e(h_1 f_1'/s_1' + \cdots + h_t f_t'/s_t') = e(1/m)$. $\square$

Lemma 3.3. *Suppose b, c are distinct residues modulo q with $(b,q) = (c,q) = 1$. Suppose that $r | \text{ord}_q b$ and for every $p^a \| r$ with $a \geqslant 1$, $p^{a+1} \nmid \text{ord}_q c$. Then there is a Dirichlet character χ modulo q such that*

$$\chi(b) = e(1/r), \qquad \chi(c)^r = 1.$$

Proof. Let $s_1 = \text{ord}_q b$ and $s_2 = \text{ord}_q c$. By Lemma 3.2, there is a character χ_1 with $\chi_1(b) = e(1/s_1)$ and therefore a character χ_2 with $\chi_2(b) = e(1/r)$. Since c has order s_2, $\chi_2(c) = e(g/s_2)$ for some integer g. Write $s_2 = vu$ where $(u,r) = 1$ and $v|r$. Define x by $xu \equiv 1 \pmod{r}$, and let $\chi = \chi_2^{xu}$. Then $\chi(b) = \chi_2(b)^{xu} = e(1/r)$ and $\chi(c) = e(gxu/s_2) = e(gx/v) = e(gx(r/v)/r)$. $\square$

Definition. *An odd number m is "good" if for every j, $1 \leqslant j \leqslant m-1$, there is a number k such that among the points $(0, k/m, kj/m) \bmod 1$, either two are equal (and not equal to the third), or two of the three distances d_1, d_2, d_3 (with sum = 1) between the points satisfy (3.1).*

Remark. To prove that a number m is good, we need only to check $2 \leqslant j \leqslant (m+1)/2$, since for $j = 1$ we take $k = 1$, and if k works for $j = j_0$ then the same k works for $j = m + 1 - j_0$.

Lemma 3.4. *Every odd prime p except $p \in P = \{3, 7, 13\}$ is good, and for $p \in P$, p^2 is good. Also, the numbers 39, 91 and 273 are good.*

Proof. A short computation implies that if $p \in P$, then p is not good, but p^2 is good. Also, by a short computation, all other odd primes $\leqslant 83$ are good, as well as 39, 91 and 273. The following j values have no associated k-value: for $m = 3$, $j = 2$; for $m = 7$, $j = 3, 5$; for $m = 13$; $j = 3, 5, 6, 8, 9, 11$; for $m = 21$, $j = 5, 17$.

Suppose that $m = p > 84$ is prime and write each product $kj = \ell p + r$ with $0 \leqslant r < p$. We shall prove that for each $j \in [2, \frac{p+1}{2}]$, there is a k so that two of the three distances satisfy $\frac{1}{3} < d_1 \leqslant d_2 < \frac{1}{2}$. We now divide up the $j \in [2, \frac{p+1}{2}]$ into 9 cases:

Case I. $j \in \{3, 5, 7, \frac{p+1}{2}\}$. For $j = 3$ take $p/6 < k < 2p/9$ and for $j = 5, 7$ take any k with $p/(2j) < k < p/(2j-2)$. There is such a k when $p > 84$. Then $p - jk$ and $jk - k$ both lie in $(p/3, p/2)$. For $j = \frac{p+1}{2}$ take $k = 2\lceil p/3 \rceil$, then $r = \lceil p/3 \rceil$, so both r and $k - r$ lie in $(p/3, p/2)$ for $p > 6$.

Case II. $9 \leqslant j < p/6 + 1$. Take $m = \left\lfloor \frac{5}{12}(j-1) \right\rfloor$. Then

$$\frac{m + 1/2}{j-1} \leqslant \frac{5}{12} + \frac{1/2}{j-1} < \frac{1}{2}, \qquad \frac{m + 1/3}{j-1} \geqslant \frac{5}{12} - \frac{7/12}{j-1} > 1/3.$$

Therefore, if

$$\frac{p(m+1/3)}{j-1} < k < \frac{p(m+1/2)}{j-1},$$

then k and $r - k$ lie in $(p/3, p/2)$. But the above interval has length $p/(6j-6) > 1$, so such a k exists.

Case III. $2 \leqslant j < p/3 + 1, j$ even. Take $k = \frac{p-1}{2}$. Then $r = p - j/2$ and both k and $r - k$ lie in $(p/3, p/2)$.

Case IV. $p/3 + 1 < j < 3p/7, j$ even. Take h so that $1 \leqslant h < \frac{p-3}{18}$ and

$$\frac{2h + 2/3}{6h+1} p < j < \frac{2h+1}{6h+1} p.$$

The largest admissible h is at least $\frac{p-19}{18}$, so the above intervals cover $(\frac{p(p-13)}{3(p-16)}, 3p/7)$, which contains $[\frac{p+4}{3}, 3p/7)$ for $p > 64$. Then take $k = \frac{p-1}{2} - 3h$, so that $r \in (p/2, 2p/3)$.

Case V. $2p/5 + 1 < j \leqslant \frac{p-1}{2}, j$ even. We take h so that $1 \leqslant h < \frac{p-3}{12}$ and

$$\frac{2h}{4h+1} p < j - 1 < \frac{2h + 1/3}{4h+1} p.$$

The largest admissible h is at least $\frac{p-13}{12}$, so these intervals cover $(2p/5, \frac{p(p-11)}{2(p-10)})$, which includes $(2p/5, \frac{p-3}{2}]$ for $p > 13$. Then take $k = \frac{p-1}{2} - 2h$, so $r - k \in (p/3, p/2)$.

Case VI. $p/3 + 1 < j \leqslant \frac{p-1}{2}, j$ odd. Take h, $0 \leqslant h < \frac{p-15}{12}$ so that

$$\frac{2h+1}{4h+3} p < j - 1 < \frac{2h + 4/3}{4h+3} p.$$

Than take $k = \frac{p-3}{2} - 2h$, so that $k - r \in (p/3, p/2)$. The above intervals cover $(p/3, \frac{p-3}{2}]$ provided that $p > 24$.

Case VII. $p/3 - 1 < j < p/3 + 1$. Write $j = \frac{p+t}{3}$, where $-2 \leqslant t \leqslant 2$, $t \neq 0$. Here we take $k = 3\lceil p/9 \rceil + b$, where $0 \leqslant b \leqslant 2$ and $t + 3b \equiv w \pmod 9$, $w \in \{5, 7\}$. If $p > 28$ then $k \in (p/3, p/2)$. If $w = 5$, then $r = 5p/9 + E$, where

$|E| \leqslant 22/9$. Thus, $r \in (p/2, 2p/3)$ when $p > 44$. When $w = 7$, $t = 1, b = 2$, then $r \in (7p/9, 7p/9 + 14/9]$.

Case VIII. $5p/21 < j < p/3 - 1, j$ odd. Take $1 \leqslant h < \frac{p-3}{18}$ so that

$$\frac{6h-1}{18h+3}p < j < \frac{2h}{6h+1}p.$$

Take $k = \frac{p-1}{2} - 3h$, so $r \in (p/3, p/2)$. The above intervals cover $(5p/21, p/3 - 1)$.

Case IX. $p/6 + 1 < j < 5p/21, j$ odd. If $p/5 < j - 1 < 4p/15$, take $k = \frac{p-5}{2}$, so that $r \in (5p/6 - 5/2, p - 5/2)$. If $p/7 < j - 1 < 4p/21$, then $k = \frac{p-7}{2}$ works and if $5p/27 < j < 2p/9$ then $k = \frac{p-9}{2}$ works. $\square$

Proof of Lemma 3.1. By hypothesis, there are two possibilities:
(i) some s_i (say s_1) is divisible by a prime power p^w other than $3, 7$, or 13;
(ii) Each s_i divides 273 and some s_i (say s_1) equals 39, 91 or 273.

Say s_1 is divisible by p^w, with $p^{w+1} \nmid s_2$ and $p^{w+1} \nmid s_3$. By Lemma 3.3, there is a character χ_1 with $\chi_1(a_2/a_1) = e(1/p^w)$ and $\chi_1(a_3/a_2) = e(m/p^w)$ for some integer m. If $p = 2$, let $\chi = \chi_1^{2^{w-1}}$, so that $\chi(a_2/a_1) = -1$ and

$$1 = \chi(a_2/a_1)\chi(a_3/a_2)\chi(a_1/a_3) = -\chi(a_3/a_2)\chi(a_1/a_3).$$

But each character value on the right is either -1 or 1, so either $\chi(a_2) = \chi(a_3)$ or $\chi(a_1) = \chi(a_3)$ and (i) is satisfied. If p is odd, let $\chi_2 = \chi_1^{p^{w-1}}$ if $p \notin P$ and $\chi_2 = \chi_1^{p^{w-2}}$ if $p \in P$. Then $\chi_2(a_2/a_1) = e(1/p^u)$, where $u = 2$ if $p \in P$ and $u = 1$ otherwise. Write $\chi_2(a_3/a_2) = e(j/p^u)$. If $j = 0$ then $\chi_2(a_2) = \chi_2(a_3)$ and (i) is satisfied. Otherwise, since p^u is good by Lemma 3.4, there is a number k so that two of the three distances of the points $(0, k/p^u, kj/p^u) \pmod 1$ satisfy (3.1). Taking $\chi = \chi_2^k$ gives (ii) for some relabeling of the a_i's.

In the case that each s_i divides 273 and $s_1 \in \{39, 91, 273\}$, by Lemma 3.3 there is a character χ_1 with $\chi_1(a_2/a_1) = e(1/r)$ and $\chi_1(a_3/a_2) = e(g/r)$ for some integer g. (here $r = s_1$). Since r is good by Lemma 3.4, there is a k such that two of the three distances of the points $(0, k/r, kj/r) \pmod 1$ satisfy (3.1). Taking $\chi = \chi_1^k$ gives (ii) for some relabeling of the a_i's. $\square$

Lemma 3.5. *Suppose that for some relabeling of a_1, a_2, a_3 and some Dirichlet character χ modulo q, $\chi(a_i) = e(r_i)$ with $0 \leqslant r_1 < r_2 < r_3 \leqslant 2$, $d_1 = r_2 - r_1$ and $d_2 = r_3 - r_2$ and (d_1, d_2) satisfies (3.1). Then there is a finite barrier $\mathcal{B}$ for $D = (q, a_1, a_2, a_3)$ with $|\mathcal{B}| \leqslant 14$. If $d_1 > \frac{1}{3}$, then $|\mathcal{B}| \leqslant 3$.*

Proof. For some $1/2 \leqslant \beta < \alpha \leqslant \sigma$ and large γ, suppose $L(s, \chi)$ has a zero at $s = \alpha + i\gamma$ of order c_1, and $L(s, \chi^2)$ has a zero at $s = \alpha + 2i\gamma$ of order c_2, where

$$(c_1, c_2) = \begin{cases} (1,2) & d_1 > \frac{1}{3} \\ (5,9) & d_1 = \frac{6}{19} \\ (3,5) & d_1 = \frac{12}{37}. \end{cases}$$

Suppose all other non-trivial zeros of L-functions modulo q have real part $\leqslant \beta$. Let

$$D_1(x) = \frac{\phi(q)\log x}{x^\alpha}(\pi_{q,a_2}(x) - \pi_{q,a_1}(x)),$$
$$D_2(x) = \frac{\phi(q)\log x}{x^\alpha}(\pi_{q,a_3}(x) - \pi_{q,a_2}(x)).$$

Let $u = \log x$. For large x, Lemma 1.1 and the identity

$$\sin(a-b) - \sin(a-c) = 2\cos(a - \tfrac{b+c}{2})\sin(\tfrac{c-b}{2})$$

give

$$\begin{aligned} D_1(x) &= \frac{4}{\gamma}\sum_{\ell=1}^{2}\frac{c_\ell}{\ell}\sin(d_1\ell\pi)\cos(\ell\gamma u - (r_1+r_2)\pi\ell) + O(1/\gamma^2), \\ D_2(x) &= \frac{4}{\gamma}\sum_{\ell=1}^{2}\frac{c_\ell}{\ell}\sin(d_2\ell\pi)\cos(\ell\gamma u - (r_2+r_3)\pi\ell) + O(1/\gamma^2). \end{aligned} \tag{3.2}$$

For $j = 1, 2$ define

$$\begin{aligned} g_j(y) &= c_1\sin(\pi d_j)\cos y + \frac{c_2}{2}\sin(2\pi d_j)\cos 2y \\ &= c_1\sin(\pi d_j)\left(\cos y + \tfrac{c_2}{c_1}\cos(\pi d_j)\cos 2y\right). \end{aligned}$$

Because $0 < d_j < 1/2$, $\cos\pi d_j$ and $\sin\pi d_j$ are both positive. We claim that

$$\min(g_1(\gamma u - (r_1+r_2)\pi), g_2(\gamma u - (r_2+r_3)\pi)) < 0 \qquad (u \geqslant 0), \tag{3.3}$$

which is equivalent to showing

$$\min\big(g_1(y), g_2(y - \pi(d_1+d_2))\big) < 0$$

for all real y. Since g_1 and g_2 are periodic and continuous, in fact the minimum above is $\leqslant -\delta$ for some $\delta > 0$. If γ is large (depending on δ), this implies that one of the two functions on the left in (3.2) is negative for all large x. Thus for large x, $\pi_{q,a_3}(x) > \pi_{q,a_2}(x) > \pi_{q,a_1}(x)$ does not occur.

To prove (3.3), we consider the one parameter family of functions $h(y;\lambda) = \cos y + \lambda\cos(2y)$ for $0 < \lambda < 1$. These are all even functions, so it suffices to look at $0 \leqslant y \leqslant \pi$. We have $h(y;\lambda)$ positive for $0 \leqslant y < v_\lambda$ and negative for $v_\lambda < y \leqslant \pi$, where $v_\lambda = \cos^{-1}[\frac{1}{4\lambda}(-1+\sqrt{8\lambda^2+1})]$. As a function of λ, v_λ decreases from $\pi/2$ at $\lambda = 0$ to $\pi/3$ at $\lambda = 1$. For $i = 1, 2$, let $z_i = v_{\lambda_i}$ for $\lambda_i = (c_2/c_1)\cos\pi d_i$. Since $\pi(d_1+d_2) < \pi$, (3.3) will follow from

$$z_1 + z_2 < \pi(d_1+d_2). \tag{3.4}$$

When $(d_1, d_2) \in \{(\frac{6}{19}, \frac{9}{19}), (\frac{12}{37}, \frac{16}{37})\}$, (3.4) follows by direct calculation. When $\frac{1}{3} < d_1$, we have $c_1 = 1$, $c_2 = 2$ and $\lambda_j = 2\cos\pi d_j$ $(j = 1, 2)$. We claim for $j = 1, 2$ that $z_j < \pi d_j$, or equivalently $\cos z_j > \cos\pi d_j = \frac{1}{2}\lambda_j$. Since $0 < \lambda_j < 1$,

$$\cos z_j = \frac{\sqrt{8\lambda_j^2+1}-1}{4\lambda_j} > \frac{\sqrt{4\lambda_j^4+4\lambda_j^2+1}-1}{4\lambda_j} = \frac{\lambda_j}{2},$$

which proves (3.4) in this case as well. □

Combining Lemmas 3.1 and 3.5 gives the following.

Corollary 3.6. *Let $s_1 = ord_q(a_2/a_1)$, $s_2 = ord_q(a_3/a_2)$ and $s_3 = ord_q(a_1/a_3)$. If one of s_1, s_2, s_3 is not in $\{3, 7, 13, 21\}$, then there is a finite barrier $\mathcal{B}$ for D with $|\mathcal{B}| \leqslant 14$.*

4. Third construction

Throughout this section, we assume that a_1, a_2, a_3 do not satisfy the conditions of Lemma 2.1.

Lemma 4.1. *Let χ be a character modulo q such that there are at least two different values among $\chi(a_1)$, $\chi(a_2)$, $\chi(a_3)$. Then the following hold:*
(a) $\chi(a_1)$, $\chi(a_2)$, $\chi(a_3)$ are distinct;
(b) $\Re\chi(a_1)$, $\Re\chi(a_2)$, $\Re\chi(a_3)$ are distinct;
(c) All the values $\chi(a_1)$, $\chi(a_2)$, $\chi(a_3)$ are not ± 1.
(d) χ has order $\geqslant 7$.

Proof. (a) If this does not hold, the conditions of Lemma 2.1 hold with $S = \{\chi\}$.

(b) If $\chi(a_1) = \overline{\chi}(a_2)$, then, by (a), $\Re\chi(a_3) \neq \Re\chi(a_1)$, and the conditions of Lemma 2.1 hold for $S = \{\chi, \overline{\chi}\}$.

(c) If $\chi(a_3) = 1$ and k is the order of the character χ, then the conditions of Lemma 2.1 hold for $S = \{\chi, \chi^2, \ldots, \chi^{k-1}\}$. If $\chi(a_3) = -1$ and none of $\chi(a_i) = 1$, then $\chi^2(a_3) = 1 \neq \chi^2(a_1)$, and the conditions of Lemma 2.1 hold for $S = \{\chi^2, \chi^4, \ldots, \chi^{2h-2}\}$ where h is the order of χ^2.

(d) This follows directly from (b) and (c).

Lemma 4.2. *There exists a character χ modulo q of order $\geqslant 7$ such that*

$$\Re(\chi(a_3) - \chi(a_2))\Re(\chi^2(a_2) - \chi^2(a_1)) \neq \Re(\chi(a_2) - \chi(a_1))\Re(\chi^2(a_3) - \chi^2(a_2)) \tag{4.1}$$

and for some integers h, k with $1 \leqslant h < k \leqslant 3$,

$$\Im(\chi^h(a_3) - \chi^h(a_2))\Im(\chi^k(a_2) - \chi^k(a_1)) \neq \Im(\chi^h(a_2) - \chi^h(a_1))\Im(\chi^k(a_3) - \chi^k(a_2)). \tag{4.2}$$

Proof. Let χ be any character modulo q such that $\chi(a_2/a_1) \neq 1$. By Lemma 4.1 (a), the values $\chi(a_1)$, $\chi(a_2)$, $\chi(a_3)$ are distinct. Denote $\chi(a_j) = e^{2\pi i \varphi_j}$ $(j = 1, 2, 3)$. By Lemma 4.1 (b), the values $\cos(\varphi_1)$, $\cos(\varphi_2)$, $\cos(\varphi_3)$ are distinct. Therefore, the matrix $A = \cos^\ell(\varphi_j)_{\ell=0,1,2}^{j=1,2,3}$ is nonsingular. Since $\cos(2\varphi) = 2\cos^2(\varphi) - 1$, the matrix $\cos(\ell\varphi_j)_{\ell=0,1,2}^{j=1,2,3}$ is also nonsingular, and this implies (4.1).

Next, by Lemma 4.1 (c), $\sin(\varphi_j) \neq 0$ $(j = 1, 2, 3)$. Therefore, the matrix $B = \sin(\varphi_j)\cos^\ell(\varphi_j)_{\ell=0,1,2}^{j=1,2,3}$ is nonsingular. Using the identities $\sin(2\varphi) = 2\sin(\varphi)\cos(\varphi)$, $\sin(3\varphi) = 2\sin(\varphi)(4\cos^2(\varphi) - 1)$, it follows that the matrix $\sin(\ell\varphi_j)_{\ell=1,2,3}^{j=1,2,3}$ is also nonsingular. This implies (4.2).

Lemma 4.3. *Let z_1 and z_2 be complex numbers. We can associate with each $\chi \in C_q$ a non-negative real number λ_χ such that*

$$
\begin{aligned}
z_1 &= \sum_{\chi \in C_q} \lambda_\chi (\overline{\chi}(a_2) - \overline{\chi}(a_1)), \\
z_2 &= \sum_{\chi \in C_q} \lambda_\chi (\overline{\chi}(a_3) - \overline{\chi}(a_2)).
\end{aligned} \tag{4.3}
$$

Proof. Write $z_j = u_j + iv_j$ $(j = 1, 2)$, where u_1, u_2, v_1, v_2 are real. By Lemma 4.2, there is a character $\chi = \chi_0$ for which (4.1) and (4.2) hold. Thus, we can find real numbers λ_1 and λ_2 such that

$$
\begin{aligned}
\lambda_1 \Re(\chi_0(a_2) - \chi_0(a_1)) + \lambda_2 \Re(\chi_0^2(a_2) - \chi_0^2(a_1)) &= u_1/2, \\
\lambda_1 \Re(\chi_0(a_3) - \chi_0(a_2)) + \lambda_2 \Re(\chi_0^2(a_3) - \chi_0^2(a_2)) &= u_2/2,
\end{aligned}
$$

and real numbers λ_3 and λ_4 such that

$$
\begin{aligned}
\lambda_3 \Im(\chi_0^h(a_2) - \chi_0^h(a_1)) + \lambda_4 \Im(\chi_0^k(a_2) - \chi_0^k(a_1)) &= v_1/2, \\
\lambda_3 \Im(\chi_0^h(a_3) - \chi_0^h(a_2)) + \lambda_4 \Im(\chi_0^k(a_3) - \chi_0^k(a_2)) &= v_2/2,
\end{aligned}
$$

By Lemma 4.1, the six characters $\chi_0, \chi_0^2, \chi_0^3, \overline{\chi}_0, \overline{\chi}_0^2, \overline{\chi}_0^3$ are distinct. Now set $\mu_\chi = \lambda_1$ for $\chi \in \{\chi_0, \overline{\chi}_0\}$, $\mu_\chi = \lambda_2$ for $\chi \in \{\chi_0^2, \overline{\chi}_0^2\}$, and $\mu_\chi = 0$ for other characters. Also, let $\nu_{\chi_0^h} = \lambda_3$, $\nu_{\overline{\chi}_0^h} = -\lambda_3$, $\nu_{\chi_0^k} = \lambda_4$, $\nu_{\overline{\chi}_0^k} = -\lambda_4$, and $\nu_\chi = 0$ for other characters. Let $\theta_\chi = \mu_\chi + \nu_\chi$ for each χ. Then (4.3) holds with $\lambda_\chi = \theta_\chi$ for each χ, but it may occur that $\theta_\chi < 0$ for some χ. However, by Lemma 4.1, $a_j \not\equiv 1 \pmod q$ for each j, so $\sum_{\chi \in C_q} \chi(a_j) = -1$ for every j. Thus, for any real y, (4.3) holds with $\lambda_\chi = \theta_\chi + y$ for each χ.

Lemma 4.4. *If a_1, a_2, a_3 do not satisfy the conditions of Lemma 2.1, then for all $\tau > 0$ and $\sigma > \frac{1}{2}$, there is a finite barrier for $D = (q, a_1, a_2, a_3)$, with each $B(\chi)$ consisting of numbers ρ with $\Re\rho \leqslant \sigma$ and $\Im\rho > \tau$.*

Proof. By Lemma 4.3, we can find such nonnegative $\nu_\chi^{(1)}$ and $\nu_\chi^{(2)}$ that

$$
\begin{aligned}
i &= \sum_\chi \nu_\chi^{(1)} (\overline{\chi}(a_2) - \overline{\chi}(a_1)), \\
-i &= \sum_\chi \nu_\chi^{(1)} (\overline{\chi}(a_3) - \overline{\chi}(a_2)), \\
i &= \sum_\chi \nu_\chi^{(2)} (\overline{\chi}(a_2) - \overline{\chi}(a_1)), \\
i &= \sum_\chi \nu_\chi^{(2)} (\overline{\chi}(a_3) - \overline{\chi}(a_2)).
\end{aligned} \tag{4.4}
$$

Fix small positive $\varepsilon > 0$ and take a positive integer Q and nonnegative integers $N_\chi^{(1)}$, $N_\chi^{(2)}$ for all characters χ modulo q such that $|\nu_\chi^{(1)} - N_\chi^{(1)}/Q| < \varepsilon$, $|\nu_\chi^{(2)} - N_\chi^{(2)}/Q| < \varepsilon$. For some $\sigma_1 \in (\beta_1, \sigma]$ and large $\gamma > \tau$, suppose that for all characters $\chi \in C_q$ and for $k = 1, 2$ the function $L(s, \chi)$ has a zero at $s = \sigma_1 + ki\gamma$ of order $N_\chi^{(k)}$. Suppose all other non-trivial zeros of L-functions modulo q have real part $\leqslant \beta_1$. Let $D_1(x) = \phi(q)(\pi_{q,a_1}(x) - \pi_{q,a_2}(x))$ and $D_2(x) = \phi(q)(\pi_{q,a_2}(x) - \pi_{q,a_3}(x))$. By Lemma 1.1 and (4.4), we have

$$\frac{\log x}{x^{\sigma_1}} D_1(x) = \frac{Q}{2\gamma}(2\cos(\gamma \log x) + \cos(2\gamma \log x) + \varepsilon_1(x) + O(1/\gamma)),$$

$$\frac{\log x}{x^{\sigma_1}} D_2(x) = \frac{Q}{2\gamma}(-2\cos(\gamma \log x) + \cos(2\gamma \log x) + \varepsilon_2(x) + O(1/\gamma)),$$

where the functions $\varepsilon_1(x)$, $\varepsilon_2(x)$ are uniformly small if ε is small. Taking into account that $\min(2\cos u + \cos 2u, -2\cos u + \cos 2u) \leqslant -1$ for all u, we obtain that for large x, $\pi_{q,a_1}(x) > \pi_{q,a_2}(x) > \pi_{q,a_3}(x)$ does not occur. □

5. A BARRIER SATISFYING GSH_q

The construction of this barrier is modeled on the construction in §2. For one character, $B(\chi)$ is infinite, the number of elements of $B(\chi)$ with imaginary part $\leqslant T$ growing like $\sqrt{T}$. By altering the parameters in the construction, we can create barriers with $\sqrt{T}$ replaced by T^ϵ for any fixed ϵ. Assume that for some relabeling of a_1, a_2, a_3, there are two characters χ_1,χ_2 satisfying

$$\chi_1(a_1) = \chi_1(a_2) \neq \chi_1(a_3), \qquad \chi_2(a_1) \neq \chi_2(a_2). \tag{5.1}$$

Suppose that $\frac{1}{2} \leqslant \beta < \sigma_2 < \sigma_1$, that t is large and that $L(s, \chi_1)$ has a simple zero at $s = \sigma_1 + it$. Suppose that $L(s, \chi_2)$ has simple zeros at the points $s = \rho_j$ $(j = 1, 2, \dots)$, where $\rho_j = \sigma_2 - \delta_j + i\gamma_j$, $\delta_j > 0$, $\gamma_j > 0$, $\delta_j \to 0$ and $\gamma_j \to \infty$ as $j \to \infty$, and $\sum 1/\gamma_j < \infty$. Also, suppose the numbers $t, \gamma_1, \gamma_2, \dots$ are linearly independent over $\mathbb{Q}$. Define

$$Z = \overline{\chi}_1(a_2) - \overline{\chi}_1(a_3), \qquad W = \overline{\chi}_2(a_2) - \overline{\chi}_2(a_1).$$

By (5.1), $Z \neq 0$ and $W \neq 0$. Also define

$$\alpha = -\frac{1}{\pi}\left(\tan^{-1}\frac{\sigma_1}{t} + \arg Z\right), \quad \beta = \frac{\arg W}{2\pi} - \frac{1}{4}.$$

Let H be the set of integers h such that $\|h\alpha + \beta\| \leqslant \frac{1}{5}$. Since the number of possibilities for Z is finite, if t is large then

$$\frac{1}{10t} \leqslant \|\alpha\| \leqslant \frac{1}{2} - \frac{1}{10t}.$$

It follows that in every set of $\lfloor 10t\rfloor+1$ consecutive integers, one of them is in H. As in section 2, define

$$D_1(x):=\phi(q)(\pi_{q,a_1}(x)-\pi_{q,a_2}(x)),\qquad D_2(x):=\phi(q)(\pi_{q,a_3}(x)-\pi_{q,a_2}(x)).$$

Suppose x is sufficiently large, and for brevity write $u=\log x$. By Lemma 1.1 and our hypotheses,

$$D_2(x)=\frac{2x^{\sigma_1}}{u}\left[\Re\left(\frac{e^{itu}}{\sigma_1+it}Z\right)+O\left(\frac{1}{u}\right)\right] \tag{5.2}$$

and

$$\begin{aligned} D_1(x)&=\frac{2x^{\sigma_2}}{u}\sum_{\gamma_j\leqslant x}\left[\Re\left(\frac{e^{(-\delta_j+i\gamma_j)u}}{\sigma_2-\delta_j+i\gamma_j}W\right)+O\left(\frac{e^{-\delta_j u}}{\gamma_j^2 u}\right)\right]+O(x^\beta\log^2 x)\\ &=\frac{2x^{\sigma_2}}{u}\sum_{\gamma_j\leqslant x}\left[\Re B_j+O\left(\frac{e^{-\delta_j u}}{\gamma_j^2}\right)\right]+O(x^\beta\log^2 x), \end{aligned} \tag{5.3}$$

where

$$B_j=W\frac{e^{(-\delta_j+i\gamma_j)u}}{i\gamma_j}. \tag{5.4}$$

By assumption, $\sum_j|B_j|\ll 1$, thus $D_1(x)\ll x^{\sigma_2}/u$. Modulo 2π,

$$\arg\frac{e^{itu}}{\sigma_1+it}Z\equiv tu-\tan^{-1}\frac{t}{\sigma_1}+\arg Z\equiv tu-\frac{\pi}{2}-\pi\alpha.$$

By (5.2), when $\|tu/\pi-\alpha\|\geqslant u^{-0.9}$, $D_2(x)\gg x^{\sigma_1}/(\log x)^{1.9}$, and thus for these x either $\pi_{q,a_3}(x)$ is the largest or smallest of the three functions. Next assume that

$$\|tu/\pi-\alpha\|\leqslant u^{-0.9}.$$

We choose δ_j and γ_j as follows: $0<\delta_j<\sigma_2-\beta$, $j^{-3}\ll\delta_j\ll j^{-3}$, $\gamma_j=2th_j+O(j^{-10})$, where for $j\geqslant 10t$ we have $h_j\in H$, $h_{j+1}>h_j$ and $j^2\leqslant h_j\leqslant j^2+j$. With these choices,

$$\sum_{j=1}^\infty\frac{e^{-\delta_j u}}{\gamma_j^2}\ll e^{-u^{1/4}}\sum_{j\leqslant u^{1/4}}1/j^4+\sum_{j>u^{1/4}}1/j^4\ll u^{-3/4}$$

and

$$\sum_{j<u^{1/4}\text{ or }j>u^{2/5}}\frac{e^{-\delta_j u}}{\gamma_j}\ll e^{-u^{1/4}}+u^{-2/5}\ll u^{-2/5}.$$

Thus, by (5.3) and (5.4),

$$D_1(x) = \frac{2x^{\sigma_2}}{u}\left[\sum_{u^{1/4}\leqslant j\leqslant u^{2/5}} \Re B_j + O(u^{-0.4})\right]. \tag{5.5}$$

Suppose $u^{1/4} \leqslant j \leqslant u^{2/5}$. Since $h_j \in H$, we have

$$\begin{aligned}\left\|\frac{1}{2\pi}\arg B_j\right\| &= \left\|\frac{1}{2\pi}\left(\arg W + \gamma_j u - \frac{\pi}{2}\right)\right\| \\ &= \left\|\beta + \frac{ut}{\pi}h_j + O(u^{-3/2})\right\| \\ &= \left\|\beta + h_j\alpha + O(u^{-0.1})\right\| \leqslant 0.21\end{aligned}$$

for large u. Hence $\Re B_j \geqslant |B_j|\cos(0.42\pi) \geqslant \frac{1}{5}|B_j|$. Therefore,

$$\sum_{u^{1/4}\leqslant j\leqslant u^{2/5}} \Re B_j \gg \sum_{u^{1/3}\leqslant j\leqslant 2u^{1/3}} \frac{1}{\gamma_j} \gg u^{-1/3}.$$

It follows from (5.5) that for u large and $\|\frac{ut}{\pi} - \alpha\| \leqslant u^{-0.9}$ that

$$D_1(x) \geqslant \frac{cx^{\sigma_2}}{(\log x)^{4/3}}$$

where $c > 0$ depends on q, t and W. This implies that the inequality $\pi_{q,a_2}(x) > \pi_{q,a_3}(x) > \pi_{q,a_1}(x)$ does not occur for large x.

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