DIVISIBILITY OF THE CENTRAL BINOMIAL COEFFICIENT $\binom{2n}{n}$

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Abstract. We show that for every fixed $\ell \in \mathbb{N}$, the set of $n$ with $n^\ell \binom{2n}{n}$ has a positive asymptotic density $c_{\ell}$, and we give an asymptotic formula for $c_{\ell}$ as $\ell \to \infty$. We also show that $\# \{n \leq x, (n, \binom{2n}{n}) = 1\} \sim cx/\log x$ for some constant $c$. We use results about the anatomy of integers and tools from Fourier analysis. One novelty is a method to capture the effect of large prime factors of integers in general sequences.

1. Introduction

That $(n + 1)|\binom{2n}{n}$ for every positive integer $n$ is a consequence of the integrality of the Catalan numbers. In [13], Pomerance raised the question of how frequently $n + k|\binom{2n}{n}$, where $k$ is a fixed integer. Pomerance showed with a simple argument that when $k$ is positive, almost all $n$ have the property $n + k|\binom{2n}{n}$, and the exceptional set up to $x$ is $O(x^{1-a_k})$ for some $a_k > 0$. When $k \leq 0$, he proved that the set of such $n$ is governed by the set of such $n$ corresponding to $k = 0$; more precisely,

$$\# \{n \leq x : (n + k)|\binom{2n}{n}\} = \# \{n \leq x : n|\binom{2n}{n}\} + O(x^{1-a_k}).$$

Pomerance conjectured that $n|\binom{2n}{n}$ on a set of positive lower density, and showed that it has upper density at most $1 - \log 2$; this is an easy consequence of the fact that if $n$ has a prime factor larger than $\sqrt{2n}$, then $n \nmid \binom{2n}{n}$. The upper asymptotic density was later improved by Sanna [14] to $1 - \log 2 - 0.0551$.

Divisibility of $\binom{2n}{n}$ by $n^\ell$ has also been considered by several people; see the On-line Encyclopedia of Integer Sequences [12], sequences A014847 ($\ell = 1$), A121943 ($\ell = 2$), A282163 ($\ell = 3$), A282346 (smallest $n > 1$ with $n^\ell |\binom{2n}{n}$, $\ell \geq 1$), A282672 ($\ell = 6$), A283073 ($\ell = 4$), and A283074 ($\ell = 5$).

Our main result is the following.

Theorem 1. Fix $\ell \in \mathbb{N}$. The set of $n$ with $n^\ell |\binom{2n}{n}$ has a positive asymptotic density $c_{\ell}$. The density may be computed as follows: Let $U_1, U_2, \ldots$ be independent uniform-[$0, 1$] random variables, and let

$$(1.1) \quad g_1 = \left[\frac{1}{U_1}\right] - 1, \quad g_2 = \left[\frac{1}{(1 - U_1)U_2}\right] - 1, \quad \ldots, \quad g_j = \left[\frac{1}{(1 - U_1)\cdots(1 - U_{j-1})U_j}\right] - 1, \ldots$$

Then

$$c_{\ell} = \mathbb{E} \prod_{j=1}^{\infty} \left(1 - 2^{-g_j} \sum_{h=0}^{\ell-1} \left(\frac{g_j}{h}\right)\right).$$

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Table 1. Numerical counts vs. theoretical limits, $1 \leq \ell \leq 3$

<table>
<thead>
<tr>
<th>Interval</th>
<th>$\ell = 1$</th>
<th>$\ell = 2$</th>
<th>$\ell = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[1, 10^5]$</td>
<td>11,360</td>
<td>193</td>
<td>1</td>
</tr>
<tr>
<td>$[1, 10^6]$</td>
<td>118,094</td>
<td>2,095</td>
<td>3</td>
</tr>
<tr>
<td>$[1, 10^7]$</td>
<td>1,211,889</td>
<td>23,921</td>
<td>67</td>
</tr>
<tr>
<td>$[1, 10^8]$</td>
<td>12,325,351</td>
<td>279,042</td>
<td>1,055</td>
</tr>
<tr>
<td>$[1, 10^9]$</td>
<td>123,795,966</td>
<td>2,994,447</td>
<td>12,968</td>
</tr>
<tr>
<td>$[1, 10^{10}]$</td>
<td>1,240,345,721</td>
<td>31,983,305</td>
<td>172,498</td>
</tr>
<tr>
<td>$(10^{17}, 10^{17} + 10^8)$</td>
<td>12,169,463</td>
<td>364,815</td>
<td>3,390</td>
</tr>
<tr>
<td>$(10^{30}, 10^{30} + 10^7)$</td>
<td>1,180,797</td>
<td>34,734</td>
<td>351</td>
</tr>
<tr>
<td>$c_\ell$</td>
<td>0.11424</td>
<td>0.0032277</td>
<td>0.000003151</td>
</tr>
</tbody>
</table>

In Table 1, we list counts for the number of $n$ in various intervals with $n^\ell \binom{2n}{n}$, $1 \leq \ell \leq 3$, and compare with the theoretical limiting densities coming from Theorem 1 (truncated to five significant decimal places). The tabulation of $n$ such that $n^\ell \binom{2n}{n}$ was performed by two programs written by the authors, one in the C language and the other in PARI-GP, the latter being slower but applicable for the larger ranges beyond $10^{17}$. The numbers for $[1, 10^k]$, $k \leq 8$, were run by both programs and agreed exactly. These counts also agree with data gathered by Giovanni Resta (personal communication), who has also provided the data for $[1, 10^{11}]$.

See Section 7 for details of the calculation of the densities and reasons why we believe the calculations to be accurate to the decimal places displayed. It is evident from Table 1 that the convergence to the limit $c_\ell$ is very slow.

**Theorem 2.** We have

$$c_\ell \sim \rho \left(2\ell + 1 - \log(2\ell \log(2\ell)) - \frac{\log \log(2\ell)}{\log 2\ell}\right),$$

as $\ell \to \infty$, where $\rho$ is the Dickman function.

The Dickman function $\rho$ is the unique continuous solution of the differential-delay equation

$$\rho(u) = 1 \quad (u \leq 1), \quad -u\rho'(u) = \rho(u - 1) \quad (u > 1).$$

Roughly, $\rho(u)$ decays like $1/\Gamma(u)$, and in fact $\rho$ is strictly decreasing for $u > 1$ and

$$\rho(u) = e^{-u(\log u + \log \log u + O(1))}.$$

Given Theorem 1, a rought heuristic for the values given in Theorem 2 is that the factor

$$1 - 2^{1-g_j} \sum_{h=0}^{\ell-1} \binom{g_j - 1}{h}$$

is close to 1 when $g_j$ is substantially larger than $2\ell$ and is close to 0 when $g_j$ is substantially smaller than $2\ell$. Thus, $c_\ell$ should be close to the probability that $g_j \geq 2\ell$ for all $j$, which equals $\rho(2\ell)$.

A related problem is the study of the set $\mathcal{B}$ of positive integers $n$ such that $n$ and $\binom{2n}{n}$ are coprime, see e.g. sequence A082916 of the OEIS [12]. In [14], Sanna showed that $\#(\mathcal{B} \cap [1, x]) \ll x/\sqrt{\log x}$ for all
x > 1. On the other hand, \(B\) contains all odd primes, and thus \(#(B \cap [1, x]) \geq (1 + o(1))x / \log x\) for all \(x \geq 2\). We sharpen these results by proving an asymptotic formula for \(#(B \cap [1, x])\).

**Theorem 3.** We have \(#\{n \leq x : (n, \binom{2n}{n}) = 1\} \sim cx/\log x\) as \(x \to \infty\), where

\[
1.4\quad c = \sum_{k=1}^{\infty} \frac{1}{k!} \int \cdots \int_{n_1 \geq 0 \forall i} h(u_1) \cdots h(u_k) \, du_1 \cdots du_{k-1}, \quad h(x) = x^{-1}2^{-1/[1/x]}.
\]

As \(h\) is bounded, the series for \(c\) converges rapidly. Numerically, \(c = 1.526453\ldots\) (See section 9). This is also a good match to numerical data, see Table 2.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(N)</th>
<th>(N/\log x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^4</td>
<td>1734</td>
<td>1.597073</td>
</tr>
<tr>
<td>10^5</td>
<td>13487</td>
<td>1.552748</td>
</tr>
<tr>
<td>10^6</td>
<td>111460</td>
<td>1.539876</td>
</tr>
<tr>
<td>10^7</td>
<td>950039</td>
<td>1.531281</td>
</tr>
<tr>
<td>10^8</td>
<td>8282970</td>
<td>1.525779</td>
</tr>
<tr>
<td>10^9</td>
<td>73631430</td>
<td>1.525883</td>
</tr>
<tr>
<td>10^{10}</td>
<td>662319904</td>
<td>1.525047</td>
</tr>
<tr>
<td>10^{11}</td>
<td>6022446576</td>
<td>1.525391</td>
</tr>
</tbody>
</table>

**Table 2.** Number, \(N\), of integers \(\leq x\) with \((n, \binom{2n}{n}) = 1\)

1.1. **Heuristics.** For most \(n\), the divisibility condition \(n^\ell \binom{2n}{n}\) is essentially determined by the largest prime factors of \(n\). By Kummer’s criterion [10], if \(p\) is prime, then \(p^\ell \binom{2n}{n}\) if and only if the addition of \(n\) and \(n\) in base-\(p\) has at least \(\ell\) carries. This is equivalent to \(\{n/p^s\} > \frac{1}{2}\) for at least \(\ell\) values of \(s \in \mathbb{N}\). If \(p\) is large, then this means (essentially) that the base-\(p\) expansion of \(n\) has at least \(\ell\) digits which are \(\geq p^{-1}/2\) (if a digit equals \(p^{-1}/2\), then it may or may not induce a carry). Supposing that \(p\|n\), the final base-\(p\) digit is zero, and the leading digit is \(< p/2\) with high probability. There are \(k = \left\lfloor \log_{p} n \right\rfloor - 1\) remaining base-\(p\) digits, and if these are randomly distributed (over all \(n \leq x\) divisible by \(p\) and not by \(p^2\)) then we expect that \(p^\ell \binom{2n}{n}\) occurs with probability close to

\[
1 - 2^{\ell-1-k} \sum_{h=0}^{\ell-k} \binom{k-1}{h}.
\]

Donelly and Grimmett [3] (see also [15]) proved that the largest prime factors of a random integer have, asymptotically, the Poisson-Dirichlet distribution. A realization of this distribution is given in terms of independent uniform-\([0, 1]\) random variables \(U_1, U_2, \ldots\). Let \((X_1, X_2, \ldots)\) be the infinite dimensional vector formed from the decreasing rearrangement of the numbers

\[
Y_1 = U_1, Y_2 = (1 - U_1)U_2, Y_3 = (1 - U_1)(1 - U_2)U_3, \ldots
\]
Then \((X_1, X_2, \ldots)\) has the Poisson-Dirichlet distribution. Let \(p_j(n)\) denote the \(j\)-th largest prime factor of \(n\). The paper [3] gives a simple, transparent proof that \((X_1, \ldots, X_k)\) and

\[
\left( \frac{\log p_1(n)}{\log n}, \ldots, \frac{\log p_k(n)}{\log n} \right)
\]

have identical distributions (asymptotically as \(x \to \infty\), where \(n\) is drawn at random from \([1, x]\)). For a discussion of other realizations of the Poisson-Dirichlet distribution, see Section 1 of [15]. Combining this with our heuristic above about divisibility of \(\binom{2n}{n}\) by \(p^\ell\), we arrive at Theorem 1.

The heuristic for Theorem 3 is simpler. If \(n\) has \(k\) prime factors \(p_1, \ldots, p_k\), with \(p_i = x^{u_i}\), then we expect \((n, \binom{2n}{n}) = 1\) with probability \(\prod_{i=1}^k 2^{1-[1/u_i]}\). Summing over all \(p_1, \ldots, p_k\) with the prime number theorem yields the result in Theorem 3.

We will make both of these heuristics precise utilizing harmonic analysis to detect the simultaneous divisibility of \(\binom{2n}{n}\) by large prime factors of \(n\). Section 3 contains the relevant estimates. In Section 2, we show that the small prime factors of \(n\) divide \(\binom{2n}{n}\) with very high probability, and can safely be ignored. We prove a result about simultaneous fractional parts of quotients of primes in Section 4 that will be needed for Theorems 1 and 3. The proof of Theorem 1 occupies Section 5 and we prove Theorem 3 in Section 6. Sections 7 and 8 are devoted to the study of the constants \(c_\ell\), culminating in the proof of Theorem 2. Finally, we describe how to compute \(c\) accurately in Section 9.

## 2. SMALL PRIME FACTORS

In this section, we will see that only the largest prime factors of \(n\) matter for Theorems 1 and 3.

**Lemma 2.1.** Let \(p\) be prime, \(v \in \mathbb{N}\), \(\ell \in \mathbb{N}\) and \(p^{\ell v} \leq x^{1/100}\). Then

\[
\# \left\{ n \leq x : p^v | n, p^{\ell v} \nmid \binom{2n}{n} \right\} \ll \frac{x^{1-\frac{1}{100}\log p}}{p^{\ell v}} e^{v/3}.
\]

**Proof.** Suppose that \(n \leq x\) and \(p^v|n\). Write \(n\) in base-\(p\) as \(n = (b_D b_{D-1} \cdots b_0)_p\), where \(D = \left\lfloor \frac{\log x}{\log p} \right\rfloor\), so that \(b_0 = \cdots = b_{v-1} = 0\). Also observe that the hypotheses imply that \(D \geq 100v\) and hence that

\[
\ell v \leq \frac{\log x}{100\log p} \leq \frac{D + 1}{100} < \frac{D}{99} \leq \frac{D - v}{98}.
\]

The number of choices for \(b_D\) is at most \(x/p^D\). By Kummer’s criterion, if \(p^{\ell v} \nmid \binom{2n}{n}\), then at most \(\ell v - 1\) of the digits \(b_v, \ldots, b_{D-1}\) are \(\geq \frac{p}{2}\). Hence, the number of choices for \((b_v, \ldots, b_{D-1})\) is at most

\[
\sum_{j=0}^{\ell v - 1} \binom{D - v}{j} \left(\frac{p - 1}{2}\right)^j \left(\frac{p + 1}{2}\right)^{D - v - j} \ll \left(\frac{p + 1}{2}\right)^{D - v}\left(\frac{D - v}{\ell v}\right).
\]

if \(p \geq 3\), and \(O(D_{\ell v})\) when \(p = 2\). Recalling that \(\ell v \leq (D - v)/98\), by Stirling’s formula we have

\[
\binom{D - v}{\ell v} \ll e^{0.057(D - v)}
\]

and thus

\[
\# \left\{ n \leq x : p^v | n, p^{\ell v} \nmid \binom{2n}{n} \right\} \ll \frac{x}{p^v} \left(\frac{e^{0.057(1 + \frac{1}{2})}}{2}\right)^{D - v} \ll \frac{x}{p^v} e^{-(D - v)/3},
\]

and the claimed inequality follows. \(\Box\)
Proposition 1. For large $x$, let $\delta$ satisfy $0 < \delta \leq 1$. For any $1 \leq n \leq x$, write $n = A_n B_n$, where $P^+(A_n) \leq x^\delta < P^-(B_n)$. Fix $\ell \in \mathbb{N}$. Then

\[ \# \left\{ n \leq x : A_n^\ell \nmid \binom{2n}{n} \right\} \ll \ell x e^{-1/(300\delta)}. \]

Proof. We may assume that $\frac{\log x}{\log 2} < \delta \leq \frac{1}{300\ell}$, else the statement is trivial. Hence, by Lemma 1,

\[ \# \left\{ n \leq x : A_n^\ell \nmid \binom{2n}{n} \right\} \ll \sum_{p \leq x^\delta} \left[ \sum_{v \leq \frac{\log x}{100\log p}} e^{v/3} p^{-v} + x e^{-1/30\log p} \right]. \]

Next, we prove analogous bounds for integers with a given smallest prime factor.

Proposition 2. The number of integer $n \leq x$ for which $(n, \binom{2n}{n}) = 1$ and $n$ has a prime factor smaller than $n^\delta$ is $O(\frac{x \log x}{\log 2} e^{-1/(3\delta)})$.

Proof. Fix $p$ and consider those $n$ with smallest prime factor $p$ and such that $p \nmid \binom{2n}{n}$. We argue as in the $\ell = 1$ case of Lemma 1, except that for fixed $b_2, \ldots, b_D$ we bound the number of possible $b_1$ such that $\sum_{j=1}^D p^{\ell_j} b_j$ has no prime factor less than $p$ with a sieve (e.g., [7, Theorem 2.2]), obtaining

\[ \#b_1 \ll \frac{p}{\log p}. \]

It follows that

\[ \# \left\{ n \leq x : n \text{ has smallest prime factor } p, \ n \nmid \binom{2n}{n} \right\} \ll \frac{x^{1-\frac{1}{30\log p}}}{p \log p}. \]

Summing over $p \leq x^\delta$ completes the proof. \qed

3. EXPONENTIAL SUM ESTIMATES

We gather together in this section various estimates for exponential sum which we will need for the proof of Theorem 1.

The first lemma is the ’Weyl-van der Corput inequality’ (see Theorems 2.2, 2.8 in [5]). It is far from the best result of its kind, but has a relatively short proof and suffices for our purposes.

Lemma 3.1. Let $j \geq 2$ be an integer, let $I$ be an interval and suppose that $f \in C^j(I)$ and that

\[ \lambda \leq |f^{(j)}(x)| \leq \alpha \lambda \]
where \( \lambda > 0, \alpha \geq 1. \) Then
\[
\sum_{n \in I} e(f(n)) \ll |I|(\alpha^2 \lambda)^{1/2} + |I|^{1 - \frac{1}{2} \alpha^{1/2}} + |I|^{1 - \frac{2}{2} \alpha^{1/2}} \lambda^{1/2},
\]
where \( J = 2^{j - 2} \).

We apply this lemma to bound a certain class of exponential sums.

**Lemma 3.2.** Let \( N \in \mathbb{N}, \) and
\[
f(u) = \alpha u + \sum_{r=r_1}^{r_2} \frac{\beta_r}{u^r},
\]
where \( \alpha \in \mathbb{R}, 1 \leq r_1 \leq r_2, \) and for some \( A \in [1, N^{1/2}] \) we have
\[
|\beta_1| \geq N^{r_1} A, \quad |\beta_r/\beta_1| \leq N^{(r-r_1)/2} (r_1 \leq r \leq r_2).
\]
Then
\[
\max_{I \subset (N, 2N)} \sum_{n \in I} e(f(n)) \ll r_2 N \left( N^{1/2} + A^{-1/4} \right),
\]
where
\[
j = 3 + \left[ \frac{\log (|\beta_1|/AN^{r_1})}{\log N} \right].
\]

**Proof.** We apply Lemma 3.1. Firstly, we may assume that \( N \) is sufficiently large and that
\[
j \leq \frac{\log \log N}{\log 2},
\]
for otherwise the conclusion is trivial. Also note that \( j \geq 3. \) Denoting by \( r^{(j)} \) the rising factorial \( r(r + 1) \cdots (r + j - 1) \), and using (3.2), we have for \( N < u \leq 2N \) the relation
\[
f^{(j)}(u) = (-1)^j \sum_{r=r_1}^{r_2} \frac{r^{(j)} \beta_r}{u^{r+j}}
\]
\[
= (-1)^j \frac{r_1^{(j)} \beta_1}{u^{r_1+j}} \left( 1 + O \left( \sum_{r=r_1+1}^{r_2} \frac{(r^{(j)})/r_1^{(j)})|\beta_r/\beta_1|}{N^{r-r_1}} \right) \right)
\]
\[
= (-1)^j \frac{r_1^{(j)} \beta_1}{u^{r_1+j}} \left( 1 + O \left( \sum_{r=r_1+1}^{r_2} \frac{(r/r_1)^j}{N^{(r-r_1)/2}} \right) \right)
\]
\[
= \left( 1 + O_{r_2} \left( N^{-1/2} \right) \right) (-1)^j \frac{r_1^{(j)} \beta_1}{u^{r_1+j}}.
\]
For large enough \( N \) it follows that
\[
\lambda \leq |f^{(j)}(u)| \leq \alpha \lambda, \quad \lambda = \frac{r_1^{(j)}|\beta_1|}{2(2N)^{r_1+j}}, \quad \alpha = 2^{r_1+j+2}.
\]
Inserting this bound into Lemma 3.1, we have
\[
\frac{1}{N} \sum_{n \in I} e(f(n)) \ll r_2 \lambda^{1/2} + N^{-1/2} + N^{-1/2} \alpha \lambda^{1/2},
\]
where \( J = 2^{j-2} \). We note that from (3.2) and the definition of \( j \),
\[
N^2 \frac{\beta_{r_1}}{AN^{r_1}} \leq N^j \leq N^3 \frac{\beta_{r_1}}{AN^{r_1}}
\]
and hence that
\[
\frac{A}{2^{r_1+j+1}N^{3}} \leq \lambda \leq r_1^{(j)} \left( \frac{A}{N^2} \right) \leq r_1^{(j)} N^{-3/2}.
\]
When \( j = 3 \), therefore, the right side of (3.5) is
\[
\ll r_2 \lambda^{1/6} + N^{-1/4} + N^{-3/4} \lambda^{-1/4} \ll N^{-1/4} + A^{-1/4}.
\]
Now assume that \( j \geq 4 \) so that \( J \geq 4 \). Then the right side of (3.5) is
\[
\ll r_2 N^{-\frac{3}{j-2}} + N^{-\frac{1}{j}} + N^{-\frac{7}{j}} (N^3)^{\frac{7}{j}} \ll r_2 N^{-\frac{1}{6}}.
\]
Combining the two cases, \( j = 3 \) and \( j > 3 \), this concludes the proof. \( \square \)

We now apply Lemma 3.2 to bound analoguous sums over primes.

**Lemma 3.3.** Assume \( f \) satisfies (3.1), where the coefficients satisfy (3.2) for some \( A \in [1, N^{1/6}] \). Then
\[
\max_{I \subseteq (N,2N]} \sum_{p \in I} e(f(p)) \ll r_2 N (\log N)^4 \left( N^{-\frac{1}{6}} + A^{-1/10} \right),
\]
where \( j \) is given by (3.3).

**Proof.** Our technique is standard. Throughout, constants implied by \( O_{--} \) and \( \ll \) may depend on \( r_1, r_2 \). We begin by applying Vaughan’s identity, taking \( U = V = N^{1/3} \) in [2, p. 139]. This gives
\[
(3.6) \quad \sum_{p \in I} e(f(p)) = O(N^{1/2}) + \sum_{n \in I} \Lambda(n) e(f(n)) = O(N^{1/2}) + S_2 + S_3 + S_4,
\]
where, following the notation from [2] (observe that \( S_1 \) is trivially zero in our case), we define
\[
S_2 = - \sum_{a \leq N^{1/3}} \Lambda(a) \sum_{b \leq N^{1/3}} \mu(b) \sum_{abc \in I} e(f(abc)),
\]
\[
S_3 = \sum_{b \leq N^{1/3}} \sum_{bc \in I} \mu(b)(\log c) e(f(bc)),
\]
\[
S_4 = \sum_{b > N^{1/3}} \sum_{bc \in I; c > N^{1/3}} \Lambda(c) e(f(bc)),
\]
where
\[
h(b) = \sum_{d | b, d > N^{1/3}} \mu(d).
\]
We may apply Lemma 3.2 directly to \( S_2 \) and to \( S_3 \); these are called “Type I” sums in the modern literature. For \( S_2 \), we fix \( a \) and \( b \) and apply Lemma 3.2 with \( N \) replaced by \( N/ab \) and \( \beta_r \) replaced by \( \beta_r/(ab)^r \). We check that
\[
A \leq N^{1/6} \leq (N/ab)^{1/2}, \quad \left| \frac{\beta_r}{\beta_{r_1}} \right| = \left| \frac{\beta_r}{\beta_{r_1}} \right| (ab)^{-(r-r_1)} \leq \left( \frac{N}{ab} \right)^{(r-r_1)/2}.
\]
Thus, for any \( a, b \) we have
\[
\sum_{abc \in I} e(f(abc)) \ll \frac{N}{ab} ((N/ab)^{-1/2} + A^{-1/4})
\]

and hence that

\[(3.7) \quad S_2 \ll N(\log^2 N)(N^{-\frac{1}{3}} + A^{-1/4}).\]

Bounding the inner sum over \(c\) in \(S_2\) is exactly analogous, where we use partial summation to remove the logarithm factor. Since \(N/b \geq N^{2/3}\), we obtain a stronger bound

\[(3.8) \quad S_3 \ll N(\log^2 N)(N^{-\frac{2}{3}} + A^{-1/4}).\]

For \(S_4\), we break up the range \(b \in (N^{1/3}, 2N^{2/3}]\) into \(O(\log N)\) dyadic intervals of the form \((B, 2B]\) where \(N^{1/3} \leq B \leq 2N^{2/3}\). Then we use Cauchy-Schwarz, followed by the trivial bound \(|h(b)| \leq \tau(b)\) to get

\[
S_4 \ll (\log N) \max_B \left( \sum_{B < b \leq 2B} h(b) \sum_{bc \in I} \Lambda(c)e(f(bc)) \right) \leq (\log N) \max_B \left( \sum_{B < b \leq 2B} h(b)^2 \right)^{1/2} \left( \sum_{B < b \leq 2B} \left| \sum_{bc \in I} \Lambda(c)e(f(bc)) \right|^2 \right)^{1/2} \ll (\log N)^{5/2} \max_B B^{1/2} \left( \sum_{B < b \leq 2B} \sum_{bc \in I} \Lambda(c)e(f(bc))^2 \right)^{1/2}.
\]

Next, we expand the square and then interchange the order of summation:

\[(3.9) \quad \sum_{B < b \leq 2B} \left| \sum_{bc \in I} \Lambda(c)e(f(bc)) \right|^2 = \sum_{\frac{N}{2R} < c_1, c_2 \leq \frac{2N}{R}} \Lambda(c_1)\Lambda(c_2) \sum_{b \in J} e(f(bc_1) - f(bc_2)),\]

where

\(J = \{B < n \leq 2B : bc_1 \in I, bc_2 \in I\}\)

is a subinterval of \((B, 2B]\). Let \(R\) be a large constant, depending on \(r_1, r_2\). The terms above with \(|c_1 - c_2| \leq \frac{RN}{BA^{1/5}}\) contribute at most \(O(N^2(\log N)^2/(A^{1/5}B))\) to the right side of (3.9). Now suppose that \(|c_1 - c_2| > \frac{RN}{BA^{1/5}}\). Write

\[f(bc_1) - f(bc_2) = ab(c_1 - c_2) + \sum_{r=r_1}^{r_2} \beta'_r, \quad \beta'_r = \beta_r \left( \frac{1}{c_1} - \frac{1}{c_2} \right).\]

We apply Lemma 3.2 with \(\beta_r\) replaced by \(\beta'_r\), \(N\) replaced by \(B\), and \(A\) replaced by

\[A' = \frac{AN^{r_1} \beta'_{r_1}}{B^{r_1} \beta_{r_1}}.\]

Since

\[|\beta'_r| \ll A \frac{|c_1 - c_2|}{c_1^{r_1}},\]

we see that

\[
\frac{\beta'_r}{\beta'_{r_1}} \ll N^{(r-r_1)/2}c_1^{-(r-r_1)} \ll N^{(r-r_1)/2}(N/B)^{-(r-r_1)} \ll B^{(r-r_1)/2}(N/B)^{-(r-r_1)/2} \ll B^{(r-r_1)/2}N^{-(r-r_1)/6},
\]

for \(r \neq r_1\).
so that the hypotheses (3.2) hold. Also, \( A' \geq A^{4/5} \) if \( R \) is large enough, and therefore
\[
\sum_{bc \in J} e(f(bc_1) - f(bc_2)) \ll B(B^{-1/2} + A^{-1/5}).
\]

Summing over all pairs \( c_1, c_2 \) we see that the expression in (3.9) is
\[
\ll \frac{N^2}{B} (\log N)^2 (N^{-1/(3-2\varepsilon)} + A^{-1/5}),
\]
and we conclude that
\[
(3.10) \quad S_{i} \ll N(\log N)^{4}(N^{-\frac{1}{3-2\varepsilon}} + A^{-1/10}).
\]
Inserting (3.7), (3.8) and (3.10) into (3.6), this completes the proof. \( \square \)

4. DETECTING FRACTIONAL PARTS

In this section we apply harmonic analysis to detect the simultaneous fractional parts of ratios of primes. Denote by \( \{ x \} \) the fractional part of \( x \).

We begin with a result of Selberg.

**Lemma 4.1.** For any \( K \in \mathbb{N} \) and any non-empty interval \( I \subset \mathbb{R}/\mathbb{Z} \), there is a trigonometric polynomial
\[
S_{K,I}^{+}(x) = \sum_{|n|\leq K} a_n e(nx)
\]
which majorizes the indicator function of \( I \) and a trigonometric polynomial
\[
S_{K,I}^{-}(x) = \sum_{|n|\leq K} b_n e(nx)
\]
which minorizes the indicator function of \( I \), and which satisfy the following:

- \( \max(|a_n|, |b_n|) \leq 4/(|n| + 1) \) for all \( n \).
- \( \int_{I} S_{K,I}(x)dx = \text{length}(I) \pm \frac{1}{K+1} \).

**Proof.** For details and explicit construction of \( S_{K,I}^{\pm} \), see Chapter 1 in [11], especially formulas (16)–(22). \( \square \)

**Definition.** A subset \( \mathcal{R} \) of \( \mathbb{R}^k \) is said to be \( t \)-simple if, for any \( 1 \leq j \leq k \) and any choice of \( z_i \in \mathbb{R} \) \((i \neq j)\), the 1-dimensional projection \( \{ z_j : (z_1, \ldots, z_k) \in \mathcal{R} \} \) consists of at most \( t \) disjoint intervals.

**Proposition 3.** Fix \( \varepsilon, \rho \) such that \( 0 < \rho < \varepsilon \) and let \( k \in \mathbb{N} \) with \( \varepsilon \leq 1/k^2 \). Suppose that \( 1 \leq m \leq x^{1/2} \), and \( M_1, \ldots, M_k \) are integers such that

- (i) \( M_i \geq x^{\varepsilon} \) for all \( i \);
- (ii) \( x/2^{k} < M_1 \cdot \cdots \cdot M_k m \leq 2x \);
- (iii) for all \( t \), \( M_i \notin \bigcup_{s \leq 1/\varepsilon+1} \{ x(1-\rho)/s, 4x/5 \} \).

Let \( \mathcal{R} \) be any \( t \)-simple subset of
\[
\{(x_1, \ldots, x_k) : M_i < x_i \leq 2M_i \ (1 \leq i \leq k), x < mx_1 \cdots x_k \leq 2x\}.
\]
and let \( \mathcal{Q} \) denote the set of all \( k \)-tuples \( q = (q_1, \ldots, q_k) \) of primes such that \( q \in \mathcal{R} \). For each \( 1 \leq j \leq k \), let \( s_j = \left\lceil \frac{\log x}{\log M_j} \right\rceil - 1 \). Then, for some \( \xi > 0 \), which depends only on \( \varepsilon, \rho \) and \( k \), we have (writing \( n = q_1 \cdots q_k m \))

\[
(4.1) \quad \#\left\{ q \in \mathcal{Q} : \forall j, q_j \mid \binom{2n}{n} \right\} = \left(1 + O(k^2 \varepsilon)\right) \prod_{j=1}^{k} \left(1 - 2^{-s_j} \sum_{h=0}^{\ell-1} \left(\begin{array}{c} s_j \\ h \end{array}\right)\right) |\mathcal{Q}| + O_{k,\varepsilon} \left(\frac{tx^{1-\xi}}{m}\right),
\]

\[
(4.2) \quad \#\left\{ q \in \mathcal{Q} : \forall j, q_j \nmid \binom{2n}{n} \right\} = \frac{1 + O(k^2 \varepsilon)}{2^{s_1 + \cdots + s_k}} |\mathcal{Q}| + O_{k,\varepsilon} \left(\frac{tx^{1-\xi}}{m}\right).
\]
\textbf{Proof.} First, we make some preliminary observations concerning the quantities \(M_j\) and \(q_j\). Let \(1 \leq j \leq k\). By (ii) and (iii), \(M_j \leq x^{1-\rho}\), hence \(s_j \geq 0\). By definition, 
\[
x^{s_j+\frac{1}{2}} < M_j \leq x^{\frac{1}{s_j+1}}.
\]
However, (i) implies that \(s_j \leq 1/\varepsilon - 1\), and hence using (iii) we in fact have stronger inequalities for \(M_j\), namely 
\[
4x^{\frac{1}{s_j+2}} \leq M_j \leq x^{\frac{1-\rho}{s_j+1}} \quad (1 \leq j \leq k).
\]
It will important for our argument below that small powers of the primes \(q_j\) stay away from \(x\); the contrary case when \(q_j^k\) is close to \(x\) for some small \(b\) and some \(j\), will be shown to be very rare in the next section.

If \(s_j = 0\) for some \(j\), then \(M_j \geq 4x^{1/2}\). But \(q_j > M_j\) and \(q_j|n\) imply that \(q_j^2 > 8n\) and hence \(q_j \uparrow \binom{2n}{n}\). Thus, the inequalities (4.1) and (4.2) follow trivially in this case.

Now assume that \(s_j \geq 1\) for every \(j\). For each \(q \in \mathbb{Q}\), let \(n = mq_1 \cdots q_k\). Since \(M_j < q_j \leq 2M_j\), (4.3) implies that \(n\) has exactly \(s_j + 2\) digits in base-\(q_j\). Moreover, the leading digit is much smaller than \(q_j/2\) since by (4.3), 
\[
\frac{n}{q_j} < \frac{2x}{M_j^{s_j+2}} \leq \frac{2}{4s_j+2} < \frac{1}{32}.
\]
Hence there are \(s_j\) base-\(q_j\) digits which could possibly induce a carry when adding \(n\) and \(n\) in base-\(q_j\). Therefore, \(\binom{2n}{n}\) is divisible by \(q_j^\ell\) if and only if for at least \(\ell\) values of \(s \in \{1, 2, \ldots, s_j\}\) we have \(n/q_j^{s+1} > 1/2\). Likewise, \(q_j \uparrow \binom{2n}{n}\) if and only if \(n/q_j^{s+1} < 1/2\) for every \(s\) in the range \(1 \leq s \leq s_j\).

Now we return to the proof of the Proposition. The number of \(q\) such that \(q_i|\) for some \(i\) is 
\[
\ll (k \log x)x^{1-\varepsilon}/m,
\]
which is negligible and can be absorbed into the error terms in (4.1) and (4.2) if \(\xi < \varepsilon\). For each \(1 \leq j \leq k\) and \(1 \leq s \leq s_j\), let \(\sigma_{j,s} \in \{0, 1\}\), and denote by \(\Sigma\) the vector of the numbers \(\sigma_{j,s}\). For each \(\Sigma\) let 
\[
Q_\Sigma := \left\{ q \in \mathbb{Q} : \left\{ \frac{mq_1 \cdots q_k}{q_j^{s+1}} \right\} \in \left[ \frac{\sigma_{j,s}}{2}, \frac{1 + \sigma_{j,s}}{2} \right] \ (1 \leq j \leq k, 1 \leq s \leq s_j) \right\}.
\]
Our main task is to prove that 
\[
|Q_\Sigma| = \frac{1 + O(k^2 \varepsilon)}{2^{s_1 + \cdots + s_k}} |Q| + O_{k,\varepsilon} \left( \frac{tx^{-1}\varepsilon}{m} \right).
\]

By our earlier remarks, the left side of (4.1) is the sum of \(Q_\Sigma\) over all \(\Sigma\) such that \(\sum_s \sigma_{j,s} \geq \ell\) for all \(j\), and the left side of (4.2) equals \(Q_\Sigma\) for the single \(\Sigma\) with \(\sigma_{j,s} = 0\) for all \(j, s\). Thus, (4.1) and (4.2) follow from (4.4).

In order to prove (4.4), fix \(\Sigma\) and apply Lemma 4.1 to the intervals \([0, 1/2]\) and \([1/2, 1]\) and with 
\[
K = \left\lfloor k \varepsilon^{-2} \right\rfloor.
\]
Define 
\[
\psi_{0,K}(x) = S_{K,[0,1/2]}^+(x) = \sum_{|n| < K} c_{0,n}^+ e(nx),
\]
\[
\psi_{1,K}(x) = S_{K,[1/2,1]}^+(x) = \sum_{|n| < K} c_{1,n}^+ e(nx).
\]
Therefore, applying Lemma 3.3, we get
\[
\sum_{q_t \in I} e(f(q_t)) \ll_k M_h (\log M_h)^4 \left( M_h^{-\frac{1}{3+\varepsilon}} + x^{-\varepsilon/4} \right) \ll x^{-\varepsilon} M_h.
\]
Summing over all \( q_i \) \((i \neq h)\), we find that \( E \ll_k x^{-1-\varepsilon} \). Combined with (4.7), this completes the proof of (4.4).

\[\square\]
5. Proof of Theorem 1

Throughout this section, we will assume that $k$ is a large integer, and that $\varepsilon, \delta$ are functions of $k$ that tend to 0 as $k \to \infty$; precisely, we take

\begin{equation}
\delta = e^{-2k/3}, \quad \varepsilon = k^{-2k}.
\end{equation}

Suppose that $x$ is a large integer. We think of $k$ being fixed and $x \to \infty$. In this section only, we adopt the following notation for functions $f(k, x)$. The notation $f(k, x) = o(g(k, x))$ means that

\[ \forall k \geq 1 : \lim_{x \to \infty} \frac{f(k, x)}{g(k, x)} = 0. \]

The notation $f(k, x) = \overline{o}(g(k, x))$ means that

\[ \lim_{k \to \infty} \limsup_{x \to \infty} \frac{f(x, k)}{g(x, k)} = 0. \]

For example, $1/k = \overline{o}(1)$ and $e^{kx^{1-1/k}} = o(x)$.

5.1. Sampling large prime factors. Take a large integer $x$, and select a random integer $n \in (x, 2x]$ with uniform probability. Following Donnelly and Grimmett [3], we select at random a $k$-tuple $q(n) = (q_1, ..., q_k)$ of prime power divisors of $n$ at random, in a size-biased fashion, together with random variables $X_1(n), \ldots, X_k(n)$. If $n$ has fewer than $k$ distinct prime factors, set $q(n) = (1, \ldots, 1)$ and $X_1(n) = \cdots = X_k(n) = 0$. Otherwise, choose $q_1|n$ at random with probability $\frac{\Lambda(q_1)}{\log n}$, where $\Lambda$ is the von Mangoldt function. For $2 \leq i \leq k$, once $q_1, \ldots, q_{i-1}$ are chosen, select $q_i|(n/q_1 \cdots q_{i-1})$ with probability $\frac{\Lambda(q_i)}{\log(n/q_1 \cdots q_{i-1})}$. Then set

\[ X_i(n) = \frac{\log q_i}{\log(n/q_1 \cdots q_{i-1})} \quad (1 \leq i \leq k) \]

We observe the relation

\begin{equation}
q_i = n^{(1-X_1(n)) \cdots (1-X_{i-1}(n))X_i(n)} \quad (1 \leq i \leq k).
\end{equation}

The following is essentially Theorem 1 of [3], although we have stated the result with a slight modification. For completeness, a proof is given in the Appendix.

Lemma 5.1. Fix $k \in \mathbb{N}$. As $x \to \infty$, the random vector $(X_1(n), \ldots, X_k(n))$ converges weakly to the uniform distribution (that is, Lebesgue measure) on $[0, 1]^k$.

We denote $\mathbb{P}_x, \mathbb{E}_x$ for the probability, respectively expectation, with respect to these random $n$, $q(n)$ and $(X_1(n), \ldots, X_k(n))$, and use $\mathbb{P}$ and $\mathbb{E}$ for the uniform probability measure on $[0, 1]^k$. For the latter, we work with independent, uniform-$[0, 1]$ random variables $U_1, \ldots, U_k$.

Definition. With $x$ fixed, let $Y_k(x)$ denote the set of $k$-tuples $y = (y_1, \ldots, y_k) \in [1, x]^k$ such that

(a) $y_i \geq x^\varepsilon$ for all $i$;

(b) $x^{1/\delta} \leq y_1 \cdots y_k \leq x^{1-\delta^2}$;

(c) for all $i$ and all $1 \leq s \leq 1/\varepsilon + 1$, $y_i \notin [x^{(1-\varepsilon^2)/s}, 8x^{1/s}]$.

Lemma 5.2. The set $Y_k(x)$ is $(1/\varepsilon + 2)$-simple.

Proof. Fix $j$ and let $y_i$ be arbitrary for $i \neq j$. Items (a) and (b) force $y_j$ into a single interval, from which are cut at most $1/\varepsilon + 1$ intervals by (c).

Lemma 5.3. We have $\mathbb{P}_x(q(n) \not\in Y_k(x) \text{ or some } q_i \text{ not prime}) = \overline{o}(1)$.
Proof. First, note that \( \mathbb{P}_x(n) \) has fewer than \( k \) prime factors. Write \( q_i = q_i(n) \) for brevity. By (5.2) and Lemma 5.1,
\[
\mathbb{P}_x(\text{some } q_i < x^\varepsilon) \leq \mathbb{P}_x(\text{some } q_i \leq n^\varepsilon) \\
\leq \mathbb{P}\left((1 - U_1) \cdots (1 - U_{i-1}) U_i \leq \varepsilon \text{ for some } i\right) + o(1) \\
\leq \mathbb{P}\left(U_i \not\in [\varepsilon^{1/k}, 1 - \varepsilon^{1/k}] \text{ for some } i\right) + o(1) \\
\leq 2k\varepsilon^{1/k} + o(1) = \overline{o}(1),
\]
upon recalling (5.1).

From (5.2), we have
\[
q_1 \cdots q_k = n^{1-(1-X_1(n)) \cdots (1-X_k(n))}.
\]
Hence,
\[
\mathbb{P}_x(x^{1-\delta} \leq q_1 \cdots q_k \leq x^{1-\delta^2}) = \mathbb{P}_x\left(\frac{\log n}{\log x} (1 - (1 - X_1(n)) \cdots (1 - X_k(n))) \in [1 - \delta, 1 - \delta^2]\right).
\]
By Lemma 5.1, as \( k \to \infty \), the variable \( 1 - (1 - X_1(n)) \cdots (1 - X_k(n)) \) converges in distribution to \( 1 - (1 - U_1) \cdots (1 - U_k) \). Now \( \mathbb{E}\log(1 - U_i) = -1 \) for each \( i \), and it follows from the Law of Large Numbers that
\[
(5.3) \quad \mathbb{P}\left((1 - U_1) \cdots (1 - U_k) \in [e^{-1.1k}, e^{-0.9k}]\right) = 1 - \overline{o}(1).
\]
Recalling the definition of \( \delta \) from (5.1), we conclude that
\[
\mathbb{P}_x(q_1 \cdots q_k \not\in [x^{1-\delta}, x^{1-\delta^2}]) = \overline{o}(1).
\]

The probability that (c) fails is at most the probability that \( n \) has a prime power factor in one of the intervals \( [x^{(1-\varepsilon^2)/s}, x^{1-\varepsilon}] \), which is easily bounded by Mertens’ theorem by
\[
\sum_{s \leq 1/\varepsilon+1} \sum_{x^{(1-\varepsilon^2)/s} < q \leq 8x^{1/s}} \frac{1}{q} \leq \frac{\varepsilon^2}{\varepsilon} = \varepsilon = \overline{o}(1).
\]
Finally, if every \( q_i \geq x^\varepsilon \) and some \( q_i \) is not prime, then \( n \) is divisible by a prime power \( p^a > x^\varepsilon \) with \( a \geq 2 \). The number of such \( n \in (x, 2x] \) is \( O(x^{1-\varepsilon^2/s}) \). This completes the proof. \( \square \)

5.2. Completing the proof. From now on, the variables \( q_i \) will denote primes. Let \( n \) and \( q(n) \) be the random quantities described above. Our main task is to show that
\[
(5.4) \quad \mathbb{P}_x\left(n^\ell \left| \begin{pmatrix} 2n \\ n \end{pmatrix} \right.\right) = c_\ell + \overline{o}(1).
\]

Theorem 1 follows immediately upon fixing \( k \), letting \( x \to \infty \), and then letting \( k \to \infty \).

We first show, using Proposition 1 and Lemma 5.3 that it suffice to consider large prime factors of \( n \) and \( q(n) \in \mathcal{Y}_k(x) \). Let
\[
B_n = \prod_{p^a \mid n, p > y} p^a,
\]
where \( y \) is the smallest power of two that is \( > x^{2\delta} \). Applying Proposition 1, followed by an application of Lemma 5.3, we see that
\[
(5.5) \quad \mathbb{P}_x\left(n^\ell \left| \begin{pmatrix} 2n \\ n \end{pmatrix} \right.\right) = \overline{o}(1) + \mathbb{P}_x\left(B_n^\ell \left| \begin{pmatrix} 2n \\ n \end{pmatrix} \right.\right) = \overline{o}(1) + \mathbb{P}_x\left(B_n^\ell \left| \begin{pmatrix} 2n \\ n \end{pmatrix} \right.\right) \text{ and } q(n) \in \mathcal{Y}_k(x).
\]
If \( q(n) \in \mathcal{Y}_k(x) \), then by (b), \( q_1 \cdots q_k \geq x^{1-\delta} \). It follows that \( B_n | q_1 \cdots q_k \), that is, \( q_1 \cdots q_k \) contains all of the large prime factors of \( n \). On the other hand, Proposition 1 implies that the probability that some prime factor \( q < y \) of \( n \) satisfies \( q^\ell \nmid \binom{2n}{n} \) is \( o(1) \). Thus

\[
P_x \left( B_n \left| \binom{2n}{n} \right. \right. \text{ and } q(n) \in \mathcal{Y}_k(x) \bigg) = P_x \left( q(n) \in \mathcal{Y}_k(x) \cap q_j \nmid \binom{2n}{n} \bigg| \binom{2n}{n} \right) (1 \leq j \leq k) + o(1).
\]

Combined with (5.5), this gives

\[
P_x \left( n^\ell \left| \binom{2n}{n} \right. \right. \right) = \overline{o}(1) + \sum_{q \in \mathcal{Y}_k(x)} P_x \left( q(n) = q_j \left| \binom{2n}{n} \right. \right. (1 \leq j \leq k).
\]

Write \( n = m q_1 \cdots q_k \). Direct computation gives

\[
P_x \left( q(n) = q_j \left| \binom{2n}{n} \right. \right. (1 \leq j \leq k) = \frac{1}{x} \sum_{x < m q_1 \cdots q_k \leq 2x} \frac{\log q_1 \cdots \log q_k}{\log n \log (n/q_1) \cdots \log (n/q_1 \cdots q_{k-1})}.
\]

It is convenient to place each \( q_i \) into a dyadic interval. For each \( i \), let \( M_i \) be the unique power of two such that \( M_i < q_i \leq 2 M_i \). By conditions (b) and (c) in the definition of \( \mathcal{Y}_k(x) \),

\[
\frac{\log q_1 \cdots \log q_k}{\log n \log (n/q_1) \cdots \log (n/q_1 \cdots q_{k-1})} = (1 + o(1)) \frac{\log M_1 \cdots \log M_k}{\log x \log (x/M_1) \cdots \log (x/M_1 \cdots M_{k-1})}.
\]

We insert this last estimate into (5.6), obtaining

\[
P_x \left( n^\ell \left| \binom{2n}{n} \right. \right. \right) = \overline{o}(1) + \frac{(1 + o(1)) \sum \log x \log \left( \frac{x}{M_1} \right) \cdots \log \left( \frac{x}{M_1 \cdots M_{k-1}} \right)}{x \log \left( \frac{x}{M_1} \right) \cdots \log \left( \frac{x}{M_1 \cdots M_{k-1}} \right)} \sum_{\substack{2^x M_i \cdots M_k, 2^x M_i \cdots M_k \leq M_i \cdots M_k \leq x \pi_1 \cdots \pi_k \leq 2 \pi_1 \cdots \pi_k \leq m q_1 \cdots q_k \leq 2x}} \sum_{q \in \mathcal{R}(\mathbf{M}, m)} 1,
\]

where the sum is taken over \( \mathbf{M} = (M_1, \ldots, M_k) \) with each \( M_i \) a power of two, and we have written \( n = q_1 \cdots q_k m \) and

\[
\mathcal{R}(\mathbf{M}, m) = \{ (z_1, \ldots, z_k) \in \mathcal{Y}_k(x) : M_i < z_i \leq 2 M_i (1 \leq i \leq k), x < m z_1 \cdots z_k \leq 2x \}.
\]

Now fix \( \mathbf{M} \) and \( m \). By Lemma 5.2, \( \mathcal{Y}_k(x) \) is \( (1/\varepsilon + 2) \)-simple and thus \( \mathcal{R}(\mathbf{M}, m) \) is also \( (1/\varepsilon + 2) \)-simple. We may then apply Proposition 3 to \( \mathcal{R}(\mathbf{M}, m) \). Condition (iii) in that Proposition holds with \( \rho = \varepsilon^2 \) on account of (c). Indeed, if

\[
M_i \in \left( x^{(1-\rho)/s}, 4 x^{1/s} \right),
\]
then

\[
q_i \in \left( x^{(1-\rho)/s}, 8 x^{1/s} \right),
\]
and (c) does not hold. Let \( s_j = \left\lfloor \frac{\log x}{\log M_j} \right\rfloor - 1 \) for each \( j \), and define

\[
F(b) = 1 - 2^{-b} \sum_{h=0}^{b-1} \binom{b}{h},
\]
By Proposition 3, we get that
\[ \sum_{q \in R(M, m), q_j \mid (\binom{2n}{n}) (1 \leq j \leq k)} 1 = (1 + O(k^2 \varepsilon)) \prod_{j=1}^{k} F(s_j) \sum_{q \in R(M, m)} 1 + O_{k, \varepsilon}(x^{1-\xi}), \]
for some \( \xi > 0 \). The final error term is negligible since the number of \( M \) is \( \ll_k (\log x)^k \). Now sum over all \( m \) and \( M \), and rewrite the final result in terms of \( q \) using (5.7) again. By (5.8) and \( O(k^2 \varepsilon) = o(1) \) we conclude that
\[ P_x \left( n \mid \left( \binom{2n}{n} \right) \right) = \bar{\sigma}(1) + (1 + \bar{\sigma}(1)) \sum_{q \in \mathcal{Y}_k(x)} P_x(q(n) = q) \prod_{j=1}^{k} F(s_j) \]
(5.9)
where (consistent with the earlier definition) by (c) we have for large enough \( x \)
\[ s_j = \left\lfloor \frac{\log x}{\log q_j} \right\rfloor - 1 \quad (1 \leq j \leq k; q \in R(M, m)). \]
(5.10)
Indeed, clearly,
\[ \left\lfloor \frac{\log x}{\log q_j} \right\rfloor \leq \left\lfloor \frac{\log x}{\log M_j} \right\rfloor, \]
and it suffices to show that
\[ \left\lfloor \frac{\log x}{\log q_j} \right\rfloor \geq s_j + 1. \]
We have \( M_j \leq x^{1/(s_j+1)} \), and next, by (c), \( M_j \leq x^{(1-\rho)/(s_j+1)} \). Hence, \( q_j \leq 2x^{(1-\rho)/(s_j+1)} \leq x^{1/(s_j+1)} \), as required for (5.10).

Using Lemma 5.3 again, followed by Lemma 5.1, we arrive at
\[ P(\ell \mid \left( \binom{2n}{n} \right) ) = \bar{\sigma}(1) + \mathbb{E}_x \prod_{j=1}^{k} F(s_j) = \bar{\sigma}(1) + \mathbb{E} \prod_{j=1}^{k} F(g_j), \]
where \( g_j \) is defined in (1.1). Finally, by the Law of Large Numbers, cf. (5.3) we have \( g_j \geq e^{j/2} \) for all \( j \geq k \) with probability \( 1 - \sigma(1) \) and this completes the proof of (5.4) upon recalling that
\[ c_\ell = \mathbb{E} \prod_{j=1}^{\infty} F(g_j). \]

6. PROOF OF THEOREM 3

The proof is similar to that of Theorem 1, but the details are simpler. In particular, we do not need the work from Section 5.1. As before, the symbols \( q \) and \( q_i \) denote primes.

For fixed \( k \in \mathbb{N} \) and \( \varepsilon > 0 \) let
\[ \mathcal{N}_{k, \varepsilon}(x) = \# \left\{ n = q_1 \cdots q_k \in (x, 2x) : \left( n, \binom{2n}{n} \right) = 1, \forall i, q_i \geq x^\varepsilon \text{ and } q_i \not\in \bigcup_{s \leq 1/\varepsilon + 1} (x^{1-\varepsilon^3}/s, 8x^{1/\varepsilon}) \right\}. \]
In contrast to the argument of the previous section, here we will take \( \rho = \varepsilon^3 \), for reasons that will become apparent later.
Lemma 6.1. For any fixed $k \geq 2$ and $\varepsilon > 0$ we have

$$|\mathcal{N}_{k,\varepsilon}(x)| = \frac{x}{\log x} \sum_{u \in [\varepsilon, 1]^k} \frac{1}{k!} \int \cdots \int h(u_1) \cdots h(u_k) \, du_1 \cdots du_{k-1} + O_k(\varepsilon^2) + O_{k,\varepsilon} \left( \frac{1}{\log x} \right),$$

where $h(v) = v^{-1}2^{1-|\varepsilon|/v}$.

Proof. Consider $n \in \mathcal{N}_{k,\varepsilon}(x)$, and write $n = q_1 \cdots q_k$ with $q_1 < \cdots < q_k$. Let

$$\mathcal{T} = \left\{ x^\varepsilon \leq y_1 < \cdots < y_k \leq x : y_1 \cdots y_k \leq 2x, \forall i : y_i \not\in \bigcup_{s \leq 1/\varepsilon + 1} \left( \frac{x(1-\varepsilon^3)/s}{s^{1/3}x^2} \right) \right\},$$

so that $q = (q_1, \ldots, q_k) \in \mathcal{T}$. For each $i$, let $M_i$ be the unique power of two such that $M_i < q_i \leq 2M_i$, and for a fixed $\mathbf{M} = (M_1, \ldots, M_k)$ let $\mathcal{T}(\mathbf{M}) = \{ \mathbf{y} \in \mathcal{T} : M_i < y_i \leq 2M_i (1 \leq i \leq k) \}$.

With $\mathbf{M}$ fixed, define $s_j = \frac{\log x}{\log M_j} - 1$. Then the hypotheses of Proposition 3 hold with $\rho = \varepsilon^3$. The set $\mathcal{T}$ is $(1/\varepsilon + 2)$-simple and hence by Proposition 3 with $m = 1$, we get that

$$|\mathcal{N}_{k,\varepsilon}(x)| = \sum_{\mathbf{q} \in \mathcal{T}(\mathbf{M})} 1 = \left( 1 + O(k^2\varepsilon^2) \right) 2^{-(s_1 + \cdots + s_k)} \sum_{\mathbf{q} \in \mathcal{T}(\mathbf{M})} 1 + O_{k,\varepsilon}(x^{1-\varepsilon}).$$

Using that $\mathcal{T}$ is $(1/\varepsilon + 2)$-simple, repeated application of the prime number theorem with classical error term implies that, for some fixed positive $c$,

$$\sum_{\mathbf{q} \in \mathcal{T}(\mathbf{M})} 1 = \int_{\mathcal{T}(\mathbf{M})} \frac{dy}{(\log y_1) \cdots (\log y_k)} + O_{k,\varepsilon}(M_1 \cdots M_k e^{-c\min_i \sqrt{\log M_i}})$$

$$= \int_{\mathcal{T}(\mathbf{M})} \frac{dy}{(\log y_1) \cdots (\log y_k)} + O_{k,\varepsilon}(xe^{-c\sqrt{\log x}}).$$

Now for any $\mathbf{y} \in \mathcal{T}(\mathbf{M})$, due to the arguments used in the previous section, we have $s_j = \frac{\log x}{\log y_j} - 1$ for each $j$. There are $\ll_k (\log x)^k$ possible tuples $\mathbf{M}$. Thus, after summing over all $\mathbf{M}$ and recalling (6.1), we obtain

$$|\mathcal{N}_{k,\varepsilon}(x)| = O_{k,\varepsilon}(x^{1-\varepsilon/2} + x/\log^5 x) + (1 + O_k(\varepsilon)) \int_{\mathcal{T}} \prod_{j=1}^k 2^{1-\frac{1}{\log y_j}} \log y_j \, dy.$$

Making the change of variables $u_i = \frac{\log y_i}{\log x}$ for each $i$, and recalling the definition of $h(\cdot)$, we see that

$$\int_{\mathcal{T}} \prod_{j=1}^k 2^{1-\frac{1}{\log y_j}} \log y_j \, dy = \int_{\mathcal{U}} h(u_1) \cdots h(u_k) x^{u_1 + \cdots + u_k} \, du_1 \cdots du_k,$$

where

$$\mathcal{U} = \left\{ \varepsilon \leq u_1 \leq \cdots \leq u_k \leq 1 : 1 \leq u_1 + \cdots + u_k \leq 1 + \frac{\log 2}{\log x}; \forall i, u_i \not\in \bigcup_{s \leq 1/\varepsilon + 1} \left[ \frac{1 - \varepsilon^3}{s}, \frac{1}{s} + \log 8 \log x \right] \right\}.$$
for some \(i \leq k\) and some \(s \leq 1/\varepsilon + 1\) has Lebesgue measure \(O(k\varepsilon^2 / \log x)\). The integrand is \(O(2^k x)\) and therefore
\[
\int \prod_{j=1}^{k} \frac{2^{1 - \log x}}{\log y_j} dy = \frac{1}{k!} \int \mathcal{V} x_{u_1 + \cdots + u_k} h(u_1) \cdots h(u_k) du_1 \cdots du_k + O\left(\varepsilon^2 x / \log x\right),
\]
where
\[
\mathcal{V} = \left\{ u \in [\varepsilon, 1)^k : 1 \leq u_1 + \cdots + u_k \leq 1 + \frac{\log 2}{\log x}\right\}
\]

Notice that in the region \(\mathcal{V}\), \(u_i \leq 1 - \varepsilon / 2\) for all \(i\) (assuming \(x \geq \exp(10/\varepsilon)\), say). Further analysis is complicated by the discontinuities of \(h(u)\) at \(u = 1/s\), \(s \in \mathbb{N}\). The function \(h()\) is, however, bounded by 2. We'll replace the function \(h\) by the continuous function \(h_\varepsilon(u)\) on \(0 \leq u \leq 1\), which equals \(h(u)\) whenever \(|u - 1/s| \geq \varepsilon^4\) for all \(2 \leq s \leq 1/\varepsilon + 1\), and otherwise is linear on each segment \([1/s - \varepsilon^4, 1/s + \varepsilon^4]\), \(2 \leq s \leq 1/\varepsilon + 1\). As before, the set of \(u \in \mathcal{V}\) that also satisfy \(|u_i - 1/s| \geq \varepsilon^4\) for some \(i\) and some \(2 \leq s \leq 1/\varepsilon + 1\) has Lebesgue measure \(O(k\varepsilon^3 / \log x)\). We thus obtain
\[
(6.3) \quad \int \prod_{j=1}^{k} \frac{2^{1 - \log x}}{\log y_j} dy = \frac{1}{k!} \int \mathcal{V} x_{u_1 + \cdots + u_k} h_\varepsilon(u_1) \cdots h_\varepsilon(u_k) du_1 \cdots du_k + O\left(\varepsilon^2 x / \log x\right).
\]

Since \(h(u)\) has bounded derivative on \([0, 1) \setminus \{1/2, 1/3, 1/4, \ldots\}\), the function \(h_\varepsilon\) satisfies
\[
|h_\varepsilon(a) - h_\varepsilon(b)| \ll \varepsilon^{-4}|a - b| \quad (a, b \in [0, 1]).
\]
Hence, letting \(v = u_1 + \cdots + u_k\), and using that \(|u_i - u_i/v| \ll 1/\log x\) for each \(i\), we get
\[
\int \mathcal{V} x_{u_1 + \cdots + u_k} h_\varepsilon(u_1) \cdots h_\varepsilon(u_k) du_1 \cdots du_k = \int \mathcal{V} x_v h_\varepsilon(u_1/v) \cdots h_\varepsilon(u_k/v) du_1 \cdots du_k + O_{k,\varepsilon}\left(\frac{x}{\log^2 x}\right)
\]

Now \(v^{k-1} = 1 + O_k(1/\log x)\). Recalling (6.3), we arrive at
\[
\int \prod_{j=1}^{k} \frac{2^{1 - \log x}}{\log y_j} dy = \frac{x}{k! \log x} \int \cdots \int h_\varepsilon(u_1) \cdots h_\varepsilon(u_k) du_1 \cdots du_k + O_{k,\varepsilon}\left(\frac{x}{\log^2 x}\right) + O\left(\varepsilon^2 x / \log x\right).
\]

We conclude by replacing each \(h_\varepsilon(u_i)\) with \(h(u_i)\). Since the set
\[
\{ u \in [\varepsilon, 1)^k : u_1 + \cdots + u_k = 1; \exists i, h(u_i) \neq h_\varepsilon(u_i)\}
\]
has \((k-1)\)-dimensional Lebesgue measure \(O(k\varepsilon^3)\), this produces an additive error term of order \(O(\varepsilon^3 x / \log x)\) (again, using that \(h_()\) and \(h_\varepsilon\) are bounded). Thus, recalling (6.1) and (6.2), the proof is complete. \(\Box\)

**Proof of Theorem 3 from Lemma 6.1.** Let \(N_k\) be the set of \(n \in [x, 2x]\) with \(k\) distinct prime factors and with \((n, \binom{2n}{n}) = 1\). Fix \(\varepsilon > 0\). Clearly
\[
N_1 \sim \frac{x}{\log x}.
\]
Now let \( k \geq 2 \). Then one of the following is true for any \( n \in \mathcal{N}_k \):

1. \( n \in \mathcal{N}_{k,\varepsilon}(x) \);
2. \( n \) has a prime factor smaller than \( x^\varepsilon \);
3. \( n \) is divisible by the square of some prime larger than \( x^\varepsilon \); or
4. \( n \) has a prime factor in \( \bigcup_{s \leq 1/\varepsilon + 1} (x^{1-\varepsilon^3}/s, 4x^{1/s}] \).

Lemma 6.1 gives the size of \( \mathcal{N}_{k,\varepsilon}(x) \). By Proposition 2, the number of \( n \) satisfying (2) is \( O(e^{-1/(3\varepsilon)}x/\log x) \). The number of \( n \) satisfying (3) is evidently \( \ll x^{1-\varepsilon}/2 \). Fixing \( s \), the number of \( n \in \mathcal{N}_k \), with all prime factors \( \geq x^\varepsilon \) and with a prime factor in \( I = (x^{1-\varepsilon}/s, 4x^{1/s}] \) is zero for \( s = 1 \), and when \( s \geq 2 \) it is at most

\[
\sum_{q_1 \in I} \sum_{\substack{q_2, \ldots, q_{k-1} \in \mathcal{P} \cap (x^{1-\varepsilon}, x] \mid q_i \nmid x^{1-\varepsilon} \forall i}} \pi \left( \frac{x}{q_1 \cdots q_{k-1}} \right) \ll \sum_{q_1 \in I} \sum_{q_2, \ldots, q_k \in (x^{1-\varepsilon}, x]} \frac{x}{\varepsilon q_1 \cdots q_{k-1} \log x} \ll \frac{x}{\log x} \frac{(\log 2/\varepsilon)^{k-1} \varepsilon^3}{\varepsilon}.
\]

After summing the above over \( s \leq 1/\varepsilon + 1 \), we see that the number of \( n \) satisfying (4) is

\[
\ll \varepsilon (\log (2/\varepsilon))^{k-1} x \frac{1}{\log x}.
\]

We conclude that

\[
|\mathcal{N}_k| = \frac{x}{\log x} \left\{ \frac{1}{k!} \int \cdots \int h(u_1) \cdots h(u_k) \, du_1 \cdots du_{k-1} + O \left( e^{-1/(3\varepsilon)} + \varepsilon (\log 2/\varepsilon)^{k-1} + o(1) \right) \right\}.
\]

The function \( h() \) is bounded above by 2, thus upon letting \( \varepsilon \to 0 \) we find that

\[
|\mathcal{N}_k| \sim \frac{x}{k! \log x} \int \cdots \int h(u_1) \cdots h(u_k) \, du_1 \cdots du_{k-1} \quad (x \to \infty)
\]

(6.4)

for each fixed \( k \). On the other hand, if \( n \) has more than \( K \) prime factors, then \( n \) has a prime factor \( < x^{1/K} \), and by Proposition 2, there are \( O(e^{-K/3}x/\log x) \) such integers. That is, for any fixed \( K \),

\[
\# \{ \mathcal{B} \cap [1, x] \} = \sum_{k=1}^{K} |\mathcal{N}_k| + O \left( e^{-K/3} \frac{x}{\log x} \right).
\]

Again using that \( h(u) \leq 2 \) for all \( u \), we see that \( |\mathcal{N}_k| \leq \frac{2^k}{(k!)^2} \frac{x}{\log x} \). Thus, letting \( K \to \infty \), Theorem 3 follows.

\[ \square \]


It is convenient here to go back to the variables \( Y_i \) given in (1.5). Moreover, in order for the product in the definition to be nonzero, we need \( Y_i \leq \frac{1}{r+1} \) for all \( i \). In particular, this shows that

\[
(7.1) \quad c_\varepsilon \leq \rho(\ell + 1) = e^{-(1+o(1))\ell \log \ell}
\]
Thus, applying (Stieltjes) integration by parts we find that
\[ \ell \rightarrow \infty, \] where \( \rho \) is the Dickman function. We have
\[ c_{\ell} = \mathbb{E} \prod_{j=1}^{\infty} g(Y_j), \quad g(y) = \begin{cases} 1 - 2^{1 - \lfloor 1/y \rfloor} \sum_{h=0}^{\ell - 1} \left( \begin{array}{c} \lfloor 1/y \rfloor - 1 \\ h \end{array} \right) & \text{if } 0 < y \leq \frac{1}{\ell + 1} \\ 0 & \text{if } y > \frac{1}{\ell + 1} \end{cases} \]

We estimate \( c_{\ell} \) using Laplace transforms. By Theorem 3.2 of [9], we have that
\[ F(s) := \int_0^{\infty} e^{-st} \left( \mathbb{E} \prod_{j=1}^{\infty} g(t Y_j) \right) dt = \frac{1}{s} \exp \left( \int_0^{\infty} \frac{g(z) - 1}{z} e^{-sz} dz \right) \quad (\Re s > 0). \]

Theorem 3.2 of [9] is only stated for real \( s > 0 \), but the proof gives the result in the full half-plane \( \Re s > 0 \). The left side of (7.3) is an entire function of \( s \in \mathbb{C} \), since
\[ \mathbb{E} \prod_{j=1}^{\infty} g(t Y_j) \leq \rho(t(\ell + 1)) \]
decays faster than exponentially in \( t \); however the right side is only well defined for \( \Re s > 0 \). We massage the right side using the standard function
\[ E_1(z) = \int_{z}^{\infty} \frac{e^{-t}}{t} dt. \]

Since \( g(z) = 0 \) for \( z > \frac{1}{\ell + 1} \) we may decompose
\[ \int_{0}^{\infty} \frac{g(z) - 1}{z} e^{-sz} dz = \int_{0}^{\frac{1}{\ell + 1}} \frac{g(z) - 1}{z} e^{-sz} dz - E_1 \left( \frac{s}{\ell + 1} \right). \]

We next use the fact that \( g(z) \) is a step-function with jumps at the points \( 1/k \), where \( k \) is an integer satisfying \( k \geq \ell + 1 \). Using the Pascal relation, and in the notation of Stieltjes integration, we have
\[ dg \left( \frac{1}{k} \right) = g \left( \frac{1}{k - 1} \right) - g \left( \frac{1}{k} \right) = -2^{1-k} \sum_{h=0}^{\ell - 1} \left( \begin{array}{c} k - 2 \\ h \end{array} \right) + 2^{1-k} \sum_{h=0}^{\ell - 1} \left( \begin{array}{c} k - 2 \\ h - 1 \end{array} \right) + \left( \begin{array}{c} k - 2 \\ h \end{array} \right) \]
\[ = -2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) \]

Thus, applying (Stieltjes) integration by parts we find that
\[ \int_{0}^{(1/\ell+1)^+} \left( g(z) - 1 \right) \frac{e^{-sz}}{z} dz = E_1 \left( \frac{s}{\ell + 1} \right) + \int_{0}^{(1/\ell+1)^+} E_1(sz) dg(z) \]
\[ = E_1 \left( \frac{s}{\ell + 1} \right) - \sum_{k \geq \ell + 1} 2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) E_1 \left( \frac{s}{k} \right). \]

Here we used that \( \lim_{y \rightarrow 0^+} g(y) = 1 \) and \( \lim_{z \rightarrow 0} E_1(sz)(g(z) - 1) = 0 \). Inserting this into (7.3) and inverting, we conclude the following:

**Proposition 4.** For any \( \sigma > 0 \), we have
\[ c_{\ell} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^s}{s} \exp \left\{ - \sum_{k \geq \ell + 1} 2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) E_1 \left( \frac{s}{k} \right) \right\} ds. \]

Computing \( c_{\ell} \) was accomplished with the Python scripts mpmath, which have a built-in function for numerically inverting the Laplace transform, and which can be computed to arbitrary precision. Table 4 shows truncated values with precision 50, 100 and 200 digits. The values for \( \ell = 1 \) are unstable in the 8th decimal place, while the calculations appear more accurate for larger \( \ell \).
from mpmath import *
mp.dps=100  # digit accuracy of internal computations
def F(s,l):
    x=mpf('0.0')
    for k in range(l+1,200):x=x+2**(1-k)*binomial(k-2,l-1)*mp.e1(s/k)
    return(mp.exp(-x)/s)
c = lambda l : mp.invertlaplace(lambda z: F(z,l),1)

TABLE 3. Python code to compute $c_\ell$

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<tr>
<th>$\ell$</th>
<th>mp.dps=50</th>
<th>mp.dps=100</th>
<th>mp.dps=200</th>
<th>scale</th>
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<tr>
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<td>0.114247430441</td>
<td>0.114247438905</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3.227780974290</td>
<td>3.22778322653</td>
<td>3.22778439553</td>
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</tr>
<tr>
<td>3</td>
<td>3.151177764641</td>
<td>3.151177748965</td>
<td>3.151177749010</td>
<td>$10^{-5}$</td>
</tr>
<tr>
<td>4</td>
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<td>1.330129946696</td>
<td>1.330129946698</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>5</td>
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<td>2.832481214761</td>
<td>2.832481214761</td>
<td>$10^{-10}$</td>
</tr>
<tr>
<td>6</td>
<td>3.403909048013</td>
<td>3.403909048013</td>
<td>3.403909048013</td>
<td>$10^{-13}$</td>
</tr>
</tbody>
</table>

TABLE 4. Values of $c_\ell$ computed by Python code with varying internal precision mp.dps

As a 2nd check, we estimated $c_\ell$ an entirely different way, using the definition of $c_\ell$ given in Theorem 1 and using Monte Carlo integration. We took $10^{10}$ random vectors of uniform-[0, 1] random variables $(U_1, \ldots, U_{50})$ and used these to estimate the expectation. The results are tabulated in Table 5. Of course, one expects deviations from the mean coming from the Central Limit Theorem. But these do appear to confirm at least the first 4 digits of the calculations in Table 4.

<table>
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<th>$\ell$</th>
<th>approximate $c_\ell$</th>
</tr>
</thead>
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<td>1</td>
<td>0.1142464511</td>
</tr>
<tr>
<td>2</td>
<td>0.0032274430</td>
</tr>
<tr>
<td>3</td>
<td>0.0000314983</td>
</tr>
</tbody>
</table>

TABLE 5. Values of $c_\ell$ computed by Monte Carlo methods, $10^{10}$ sample vectors

8. PROOF OF THEOREM 2

We use Proposition 4 and invert using the saddle-point method, as in §III.5 of [16]. By the shape of the binomial distribution, $g(z)$ transitions from being close to 1 to being very small in the vicinity of $z = \frac{1}{2\ell}$. Recall the definition (7.4) of $E_1(z)$ and define

\begin{equation}
Ein(s) := \gamma + \log s + E_1(s) = \int_0^s \frac{1-e^{-t}}{t} \, dt,
\end{equation}

which is an entire function of $s$; see [16, Theorem 5.9, §III.5] for a proof of the two representations in (8.1). By [16, Theorem 5.10, §III.5], we have

\begin{equation}
\hat{\rho}(s) := \int_0^\infty \rho(t)e^{-ts} \, dt = e^{\gamma - Ein(s)}.
\end{equation}
To bound the integral in Proposition 4, we define

\[
J(w, u) := \sum_{k=\ell+1}^{\infty} 2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) \left( E_1(w) - E_1 \left( \frac{wu}{k} \right) \right) = E_1(w) - \sum_{k=\ell+1}^{\infty} 2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) E_1 \left( \frac{wu}{k} \right).
\]

In this notation, plus (8.1), Proposition 4 implies that

\[
c_\ell = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^s \exp \left\{ \gamma - \Ein(s/u) + J(s/u, u) \right\} ds
\]

\[
= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{uw} \exp \left\{ \gamma - \Ein(w) + J(w, u) \right\} dw,
\]

where \( u \geq 1 \) is an arbitrary parameter, to be chosen later to make \( J(s/u, u) \) small when \( s \approx \sigma \).

Comparing (8.4) with (8.2), we will see that the optimal choice of \( u \) is very close to the optimal value needed to compute \( \rho(u) \) by inverting \( \hat{\rho} \), namely

\[
\sigma = -\xi_0 := -\xi(u),
\]

where \( \xi = \xi(u) \) satisfies \( e^\xi = 1 + u\xi \). We note that

\[
\xi(u) = \log(u \log u) + \frac{\log \log u}{\log u} + O \left( \frac{(\log \log u)^2}{\log^2 u} \right).
\]

We record estimates for \( \hat{\rho}(s) \) on vertical segments from [16, Lemma 5.12, Ch. III].

**Lemma 8.1.** Let \( u \geq 1 \) and \( \xi = \xi(u) \). For \( w = -\xi + i\tau \), we have

\[
\hat{\rho}(w) = e^{\gamma - \Ein(w)} = \begin{cases} 
O \left( \exp \left\{ -\Ein(-\xi) - \frac{\tau^2 u}{2\pi^2} \right\} \right) & \text{if } |\tau| \leq \pi \\
O \left( \exp \left\{ -\Ein(-\xi) - \frac{u}{\pi^2 + \tau^2} \right\} \right) & \text{if } |\tau| > \pi \\
\frac{1}{w} \left( 1 + O \left( \frac{1 + u\xi}{|w|} \right) \right) & \text{if } |\tau| > 1 + u\xi.
\end{cases}
\]

We also use a standard bound for the binomial distribution which follows quickly, for example, from Hoeffding's inequality applied to Bernoulli random variables \( X_i \) with \( \mathbb{P}(X_i = 1) = \frac{1}{2} \).

**Lemma 8.2.** We have

\[
2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) \ll \exp \left\{ -\frac{(k - 2\ell)^2}{2k} \right\}.
\]

**Lemma 8.3.** Let \( A_\ell \) be the random variable with

\[
\mathbb{P}(A_\ell = k) = a_{k, \ell} := 2^{1-k} \left( \frac{k - 2}{\ell - 1} \right) \quad (k \geq \ell + 1).
\]

Then, for \( \ell \geq 4 \) we have

(a) \( \mathbb{E}A_\ell = 2\ell + 1; \)

(b) \( \mathbb{E}|A_\ell - 2\ell|^B \ll_B \ell^{B/2} \) for all \( B \geq 0; \)

(c) \( \mathbb{E}A_\ell^{-1} = \frac{1}{2\ell} + O \left( \frac{1}{\ell^3} \right) \)

(d) \( \mathbb{E}A_\ell^{-2} = \frac{1}{4\ell^2} + \frac{1}{8\ell^3} + O \left( \frac{1}{\ell^4} \right) \)

(e) \( \mathbb{E}A_\ell e^{z/A_\ell} \ll \ell e^{z/(2\ell)} \) uniformly for \( 0 \leq z \leq \ell^{4/3} \).

**Remark.** The random variables are well-defined since \( \sum_k \mathbb{P}(A_\ell = k) = g(0^+) - g(1/\ell) = 1. \)
Proof. Identity (a) follows from
\[ \mathbb{E}A_\ell = 1 + \mathbb{E}(A_\ell - 1) = 1 + \sum_k (k - 1)a_{k,\ell} = 1 + 2\ell \sum_k a_{k,\ell+1} = 2\ell + 1. \]
The estimate (b) follows from Lemma 8.2:
\[ \mathbb{E}|A_\ell - 2\ell|B \ll \sum_{k > \ell} |k - 2\ell|B e^{-\frac{1}{2\pi}(k-2\ell)^2} \ll \ell^B/2. \]
We prove (c) and (d) in a manner similar to that of the proof of (a). First, for \( k \geq 4 \) we have
\[ \frac{1}{k} = \frac{1}{k - 2} - \frac{2}{(k - 2)(k - 3)} + O \left( \frac{1}{k^3} \right) \]
and thus
\[ \mathbb{E}A_{\ell}^{-1} = O \left( \frac{1}{\ell^4} \right) + \sum_k \left( \frac{1}{k - 2} - \frac{2}{(k - 2)(k - 3)} \right) a_{k,\ell} \]
\[ = O \left( \frac{1}{\ell^4} \right) + \frac{1}{2(\ell - 1)} \sum_k a_{k,\ell-1} - \frac{2}{4(\ell - 1)(\ell - 2)} \sum_k a_{k,\ell-2} \]
\[ = \frac{\ell - 3}{2(\ell - 1)(\ell - 2)} + O \left( \frac{1}{\ell^4} \right) = \frac{1}{2\ell} + O \left( \frac{1}{\ell^4} \right). \]
Similarly,
\[ \mathbb{E}A_{\ell}^{-2} = \sum_{k \geq \ell + 1} a_{k,\ell} \left( \frac{1}{(k - 2)(k - 3)} - \frac{5}{(k - 2)(k - 3)(k - 4)} + O \left( \frac{1}{k^4} \right) \right) \]
\[ = O \left( \frac{1}{\ell^4} \right) + \frac{1}{4(\ell - 1)(\ell - 2)} \sum_k a_{k,\ell-2} - \frac{5}{8(\ell - 1)(\ell - 2)(\ell - 3)} \sum_k a_{k,\ell-3} \]
\[ = \frac{2\ell - 11}{8(\ell - 1)(\ell - 2)(\ell - 3)} + O \left( \frac{1}{\ell^4} \right) \]
\[ = \frac{1}{4\ell^2} + \frac{1}{8\ell^3} + O \left( \frac{1}{\ell^4} \right). \]
Finally we prove part (e) using Lemma 8.2. Let \( k_0 = \lfloor 2\ell - 10\ell^2/3 \rfloor \) and \( k_1 = 4\ell \). We have
\[ \mathbb{E}A_\ell e^{z/A_\ell} \ll \ell e^{z/k_0} + \ell \sum_{k = k_0 + 1}^{2\ell} \exp \left\{ - \frac{(2\ell - k)^2}{2k} + \frac{z}{k} \right\} + \ell \sum_{k > 10\ell} \exp \left\{ - \frac{(k - 2\ell)^2}{2k} + \frac{z}{k} \right\} \]
\[ \ll \ell e^{z/(2\ell)} + \ell \sum_{k = k_0 + 1}^{2\ell} e^{-\ell^3/3} + \ell \sum_{k = k_1}^{\infty} e^{-k/8 + z/k_1} \]
\[ \ll \ell e^{z/(2\ell)}, \]
as required. \( \square \)

We use the previous two lemmas to estimate \( J(w, u) \), as defined in (8.3).

**Proposition 5.** Suppose that \( u = 2\ell + O(\log \ell) \) and \( \xi = \xi(u) \). Then, on the vertical line \( \Re w = -\xi \) we have the crude bound
\[ \mathcal{J}(w, u) \ll \frac{e^{\xi}}{|w|} \ll \frac{\ell \log \ell}{|w|}. \]
**Furthermore, if** \(|w| < \ell^{1/4}\) **then we have the asymptotic**

\[
J(w, u) = e^{-w} \left[ \frac{u - w - 1}{2\ell} - 1 + O(|w|^2 \ell^{-3/2}) \right].
\]

**Proof.** Using integration by parts, we see that

\[
E_1(w) - E_1 \left( \frac{wu}{k} \right) = \int_1^{u/k} \frac{e^{-wz}}{z} \, dz \quad \Rightarrow \quad \frac{e^x - e^{-wu/k} (k/u)}{w} - \frac{1}{w} \int_1^{u/k} \frac{e^{-wz}}{z^2} \, dz \ll \frac{e^x + e^{xu/k} (k/u)}{|w|} + \frac{(k/u) \max(e^x, e^{xu/k})}{|w|} \ll \frac{(e^x + e^{xu/k})(1 + k/u)}{|w|}.
\]

Apply (8.3), followed by an application of Lemma 8.3 (a) and (e). We have

\[
J(w, u) \ll \frac{1}{|w|} \sum_{k=\ell+1}^{\infty} 2^{1-k} \left( \frac{e^x + e^{xu/k}}{1} \right) (1 + k/u) \ll \frac{1}{|w|} \mathbb{E} \left( 1 + A\ell/u \right) \left( e^x + e^{xu/A\ell} \right) \ll \frac{\mathbb{E} A\ell (e^x + e^{xu/A\ell})}{u|w|} \ll \frac{\ell e^x + \ell e^{xu/(2\ell)}}{\ell|w|},
\]

and (8.7) follows from the bounds on \(u\).

Now suppose that \(|w| < \ell^{1/4}\). By (8.6), (8.9) and Lemma 8.2, the terms in the definition (8.3) of \(J(w, u)\) corresponding to \(|k - 2\ell| > 100(\ell \log \ell)^{1/2}\) have total sum

\[
\ll \frac{e^{2x}}{|w|} \sum_{|k-2\ell|>100(\ell \log \ell)^{1/2}} (1 + k/u)a_{k,\ell} \ll \frac{1}{\ell^{100}}.
\]

When \(|k - 2\ell| < 100(\ell \log \ell)^{1/2}\), the fraction \(u/k = 1 + O(\sqrt{\log \ell/\ell})\). Hence

\[
E_1(w) - E_1 \left( \frac{wu}{k} \right) = e^{-w} \int_0^{u-1} \frac{e^{-wv}}{1 + v} \, dv \ll e^{-w} \int_0^{u-1} \left( 1 - (w + 1)v + O(|w|^2 v^2) \right) \, dv \ll -e^{-w} \left[ 1 - \frac{u}{k} + (w + 1) \left( 1 - \frac{u}{k} \right)^2 + O \left( |w|^2 \frac{|k - u|^3}{\ell^3} \right) \right].
\]

By Lemma 8.3 (b),

\[
\mathbb{E}|k - u|^3 \ll \mathbb{E}|k - 2\ell|^3 + |2\ell - u|^3 \ll \ell^{3/2}
\]
and thus the big-$O$ term above is $\ll |w|^2 \ell^{-3/2}$. Reintroducing the summands $|k - 2\ell| \geq 100(\ell \log \ell)^{1/2}$, which are negligible by (8.10), we find using Lemma 8.3 (c) and (d) that

$$J(w, u) = O\left(\frac{1}{\ell^{100}}\right) - e^{-w}\left[1 - uA_{\ell}^{-1} + (w + 1)\frac{1 - u}{A_{\ell}} + O(|w|^2 \ell^{-3/2})\right]$$

$$= O\left(\frac{1}{\ell^{100}}\right) - e^{-w}\left[1 - u + (w + 1)\left(1 - \frac{u}{2\ell}\right)^2 + \frac{u^2}{8\ell^3}\right] + O(|w|^2 \ell^{-3/2})$$

$$= e^w\left[u - w - \frac{1}{2\ell} - 1 + O(|w|^2 \ell^{-3/2})\right].$$

Here we used repeatedly the bounds $|w| \geq 1$ and $|u - 2\ell| \ll \log \ell$. This completes the proof of (8.8). □

We now complete the proof of Theorem 2. Begin with the $w$-integral on the right side of (8.4) and define

$$u = 2\ell + 1 - \xi(2\ell), \quad \sigma = u\xi(u).$$

Since

$$\xi'(u) = \frac{\xi + 1}{u(\xi - 1) + 1} \ll \frac{1}{u},$$

and $\xi(2\ell) \ll \log \ell$, it follows that

$$\xi(2\ell) = \xi(u) + O\left(\frac{\log \ell}{\ell}\right)$$

and hence that

$$u = 2\ell + 1 - \xi(u) + O\left(\frac{\log \ell}{\ell}\right).$$

Plugging this into (8.8), we see that when $w = -\xi + i\tau$ and $|\tau| < \ell^{1/4}$, we have the bound

$$(8.12) \quad J(-\xi + i\tau, u) = e^{-w\left(-\frac{i\tau}{2\ell} + O(|w|^2 \ell^{-3/2})\right)} \ll |\tau| \log \ell + \frac{\log^3 \ell + |\tau|^2 \log \ell}{\ell^{1/2}} \quad (|\tau| < \ell^{1/4}).$$

We now insert the estimates (8.12), (8.7) and the bounds from Lemma 8.1 into the right side of (8.4). Let

$$\tau_1 = 100\sqrt{\frac{\log u}{u}}, \quad \tau_2 = \pi, \quad \tau_3 = 1 + u\xi(u).$$

Write $w = -\xi + i\tau, \xi = \xi(u)$.

Our first task is to show that the part of the integral with $|\tau| > \tau_1$ is negligible. When $\tau_1 \leq |\tau| \leq \tau_2$, Lemma 8.1 and (8.12) imply that

$$e^{\gamma - \text{Ein}(w) + J(w,u)} \ll e^{-\text{Ein}(-\xi) - \tau^2/2\pi^2 + O(|\tau| \log \ell)}$$

$$\ll e^{-\text{Ein}(-\xi) - 1000 \log u}.$$ 

When $\tau_2 \leq |\tau| \leq \tau_3$, Lemma 8.1, (8.7) and (8.12) together imply

$$e^{\gamma - \text{Ein}(w) + J(w,u)} \ll e^{-\text{Ein}(-\xi) - \frac{\tau^2}{2\pi^2 \xi^2} + O(\ell^{3/4} \log \ell)}$$

$$\ll e^{-\text{Ein}(-\xi) - \frac{\tau^2}{u^2 \pi^2}},$$

and when $|\tau| > \tau_3$, Lemma 8.1 and (8.7) give

$$e^{\gamma - \text{Ein}(w) + J(w,u)} = \frac{1}{w} \left(1 + O\left(\frac{\ell \log \ell}{|w|}\right)\right).$$
We find that the portion of the $w$-integral in (8.4) corresponding to $|\tau| \geq \tau_1$ is
\[
\ll \frac{e^{-u\xi - \operatorname{Ein}(-\xi)}}{\ell^{500}} + e^{-u\xi} \int_{\tau_1}^{\infty} \left| \frac{e^{iu\tau}}{\tau} \left( 1 + O \left( \frac{\ell \log \ell}{\tau} \right) \right) \right| d\tau
\ll \frac{e^{-u\xi - \operatorname{Ein}(-\xi)}}{\ell^{500}} + e^{-u\xi}
\ll \frac{e^{-u\xi - \operatorname{Ein}(-\xi)}}{\ell^{500}},
\]
upon appealing to the easy bound $-\operatorname{Ein}(-\xi) \gg \xi^{-1}e^{\xi} \gg \ell$.

Finally, we consider $|\tau| \leq \tau_1$. By Lemma 8.1 and (8.7) it follows that
\[
\frac{1}{2\pi i} \int_{-\xi-i\tau_1}^{\xi+i\tau_1} e^{uw} e^{\gamma - \operatorname{Ein}(w) + J(w,u)} dw = K(u) + O \left( e^{-u\xi - \operatorname{Ein}(-\xi)} \frac{\log^2 \ell}{\ell} \right),
\]
where
\[
K(u) = \frac{1}{2\pi i} \int_{-\xi-i\tau_1}^{\xi+i\tau_1} e^{uw} e^{\gamma - \operatorname{Ein}(w)} dw.
\]
Extending the limits to $-\xi \pm i\infty$ produces a small error term by Lemma 8.1 and it follows from (8.2) that
\[
\rho(u) - K(u) \ll \int_{|\tau|>\tau_1} |e^{uw - \operatorname{Ein}(w)}| dw \ll \frac{e^{-u\xi - \operatorname{Ein}(-\xi)}}{\ell^{100}}.
\]
Gathering these estimates together, we deduce that
\[
c_l = \rho(u) + O \left( \frac{\log^2 \ell}{\ell} e^{-u\xi - \operatorname{Ein}(-\xi)} \right).
\]
By Theorem 5.13 of [16, Ch. III], we have
\[
(8.13) \quad \rho(u) = \left( 1 + O \left( \frac{1}{u} \right) \right) \left( \frac{\xi}{2\pi(u(\xi - 1) + 1)} \right)^{1/2} e^{\gamma - u\xi - \operatorname{Ein}(-\xi)} \gg \frac{1}{u^{1/2}} e^{-u\xi - \operatorname{Ein}(-\xi)}
\]
and thus
\[
(8.14) \quad c_l = \rho(u) \left( 1 + O \left( \frac{\log^2 \ell}{\ell^{1/2}} \right) \right).
\]
Finally, we estimate the error made by replacing $u$ by
\[
u^* = 2\ell + 1 - \log(2\ell \log(2\ell)) - \frac{\log \log(2\ell)}{\log 2\ell}
\]
in (8.14). By (8.6),
\[
|u - u^*| \ll \frac{(\log \log \ell)^2}{\log^2 \ell}.
\]
Hence, using (8.13), (8.6), the bound $\xi'(u) \ll 1/u$ and the bounds
\[
\operatorname{Ein}(-\xi(u)) - \operatorname{Ein}(-\xi(u^*)) \ll \frac{e^{\xi(u)}}{\xi(u)} |\xi(u^*) - \xi(u)| \ll |u - u^*|,
\]
\[
u\xi(u) - u^*\xi(u^*) \ll |u - u^*| \log u,
\]
we see that
\[
\rho(u) \sim \rho(u^*) \quad (u \to \infty).
\]
Combining this with (8.14), this completes the proof of Theorem 2.
9. Numerical Computation of \( c \)

The terms with \( k = 1 \) and \( k = 2 \) in (1.4) contribute 1, respectively. \( \sum_{m=2}^{\infty} 2^{1-m} \log \left( \frac{m}{m-1} \right) = 0.50783922868438392189041 \ldots \). Define
\[
 f(t) = \sum_{k=3}^{\infty} \frac{1}{k!} \int \cdots \int h(u_1) \cdots h(u_k) \, du_1 \cdots du_{k-1},
\]
so that \( c = f(1) + 1.50783922868438392189041 \ldots \). Extend the definition of \( h \) to \((0, \infty)\) by defining \( h(u) = 1/u \) for \( u \geq 1 \). In this way, \( h(u) = 1/u \) for \( u > 1/2 \), and thus \( h \) is \( C^\infty \) near \( t = 1 \). As in previous sections, define the Laplace transform
\[
 F(s) = \int_0^{\infty} f(t) e^{-st} \, dt = e^{-t} - 1 - J^2/2. \quad J = \int_0^{\infty} h(u) e^{-su} \, du.
\]
Using that \( h(u) = u^{-1}2^{1-m} \) for \( \frac{1}{m+1} < u \leq \frac{1}{m}, m \geq 1 \), and recalling the definition (7.4) of \( E_1(z) \), we quickly derive
\[
 \int_0^{\infty} h(u) e^{-su} \, du = \sum_{m=1}^{\infty} 2^{1-m} \int_{1/(m+1)}^{1/m} \frac{e^{-su}}{u} \, du + \int_1^{\infty} \frac{e^{-su}}{u} \, du = \sum_{m=2}^{\infty} 2^{1-m} E_1(s/m).
\]
Again, we use the Python package \texttt{mpmath} to numerically invert the Laplace transform \( F(s) \), and this gives \( c = f(1) = 1.526453 \ldots \).

Appendix A. Proof of Lemma 5.1

Recall that for random \( q = q(n) = (q_1, \ldots, q_k) \) we defined
\[
 X_i(n) = \frac{\log q_i}{\log \left( \frac{n}{q_1 \cdots q_{i-1}} \right)}.
\]
It suffices to show that for any real numbers \( 0 < a_i < b_i < 1 \) \((1 \leq i \leq k)\),
\[
 \mathbb{P}_x(a_i \leq X_i(n) \leq b_i \,(1 \leq i \leq k)) \rightarrow \prod_{i=1}^{k} (b_i - a_i) \quad (x \to \infty).
\]
Below, constants implied by \( O - \) an \( \ll - \) may depend on \( k \) and the \( a_i, b_i \). From (5.2), if \( X_i \leq b_i \) for all \( i \) then
\[
 \frac{n}{q_1 \cdots q_{i-1}} \gg n^{(1-b_1) \cdots (1-b_{i-1})}.
\]
Hence, writing \( c = (1 - b_1) \cdots (1 - b_k) \min_i a_i \), we have \( q_i > n^c \) for all \( i \) under the assumption that \( a_i < X_i(n) < b_i \) for every \( i \). If some \( q_i \) is not prime, then \( n \) is divisible by a prime power \( p^a > x^{c/2}/ \log x \) with \( a \geq 2 \) and the number of such \( n \in (x, 2x) \) is \( O(x^{-c/2}) \). Thus, we may assume that the \( q_i \) are all prime. In this case, \( \log q_i \Lambda(q_i) \) and hence \( X_i(n) \) equals the probability that \( q_i \) is chosen at step \( i \). We calculate, using (A.1),
\[
 \mathbb{P}_x(a_i \leq X_i(n) \leq b_i \,(1 \leq i \leq k)) = \frac{1}{x} \sum_{x < n \leq 2x} \left( \sum_{q_1|n} \cdots \sum_{q_k|n} X_1(n) \cdots X_k(n) \right) \sum_{a_1 \leq X_1(n) \leq b_1} \sum_{a_1 \leq X_k(n) \leq b_1} X_k(n).
\]
On the right side, the variables \( q_i \) are no longer random, but we still define \( X_i(n) \) by (A.1). Since \( \log x \leq \log n \leq \log(2x) \), the above expression is bounded below by

\[
(1 + O(1/\log x)) \sum_{a_1 \log(2x) \leq \log q_1 \leq b_1 \log x} \frac{\log q_1}{q_1} \ldots \sum_{a_k \log(q_1 \ldots q_{k-1}) \leq \log q_k \leq b_k \log(\frac{x}{q_1 \ldots q_{k-1}})} \frac{\log q_k}{\log(\frac{x}{q_1 \ldots q_{k-1}})},
\]

and bounded above by the same expression with “\( x \)” and “\( 2x \)” interchanged in the logarithms.

For each fixed \( q_1, \ldots, q_{i-1} \), Mertens’ estimate gives

\[
\sum_{a_i \log(\frac{x}{q_1 \ldots q_{i-1}}) + O(1) \leq \log q_i \leq b_i \log(\frac{x}{q_1 \ldots q_{i-1}}) + O(1)} \frac{\log q_i}{\log(\frac{x}{q_1 \ldots q_{i-1}})} = b_i - a_i + O\left(\frac{1}{\log x}\right),
\]

and the desired result (A.2) follows.

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