THE PRIME NUMBER RACE AND ZEROS OF L-FUNCTIONS OFF THE CRITICAL LINE, II

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Abstract. We continue our examination the effects of certain hypothetical configurations of zeros of Dirichlet L-functions lying off the critical line on the relative magnitude of the functions $\pi_{q,a}(x)$. Here $\pi_{q,a}(x)$ is the number of primes $\le x$ in the progression $a \mod q$. In particular, we look at situations where $\pi_{q,1}(x)$ is simultaneously greater than, or simultaneously less than, each of k functions $\pi_{q,a_i}(x)$ $(1 \le i \le k)$. We also consider the total number of possible orderings of r functions $\pi_{q,a_i}(x)$ $(1 \le i \le r)$.

1. Introduction

Denote by $\pi_{q,a}(x)$ the number of primes $p \leqslant x$ with $p \equiv a \pmod q$. This paper is a continuation of our investigations from [FK1] on problems concerning the relative magnitude of $\pi_{q,a}(x)$ for a fixed q and varying a. More about the background of the "prime race" problems may be found in [FK1] and [FK2]. As in [FK1] we are concerned with the consequences of hypothetical configurations of zeros of Dirichlet L-functions lying off the critical line. Roughly speaking, each zero of an L-function imparts an oscillation on the functions $\pi_{q,a}(x)$, the zeros with largest real part giving the largest oscillations. In [FK1] we were concerned with the orderings of three functions $\pi_{q,a_i}(x)$ $(i = 1, 2, 3)$ which occur for arbitrarily large x. Let F_q^* denote the multiplicative group of reduced residues modulo q. Our principal result, in simple terms, was that for all $q \geqslant 5$ and distinct $a_1, a_2, a_3 \in F_q^*$, there are finite configurations of hypothetical zeros which, if they really existed, would imply that one of the orderings does not occur for large x. Also, configurations can be constructed so the zeros all have imaginary parts $\geqslant \tau$ for any given $\tau > 0$. The point of the exercise is this. If one wishes to prove that all 6 orderings of the functions occur for arbitrarily large x, one must prove in particular that our hypothetical configurations are not possible.

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In this paper we address two main types of problems. First, if D is a subset of $F_q^*\backslash\{1\}$, can it occur for arbitrarily large x that $\pi_{q,1}(x)$ is simultaneously smaller than, or simultaneously large than, each function $\pi_{q,a}(x)$ ($a \in D$) ? Secondly, given a subset D of F_q^*, how many of the $|D|!$ possible orderings of the functions $\pi_{q,a}(x)$ ($a \in D$) occur for arbitrarily large x ? In the language of Knapowski and Turán, consider a game with players $a_1, \dots, a_k$, player a_i having a score of $\pi_{q,a_i}(x)$ at time x. Our questions can then be phrased as (i) Does player 1 lead infinitely often or trail infinitely often? (ii) How many of the $|D|!$ orderings of the players occur infinitely often?

Throughout, q is a natural number, $q \geqslant 3$. Below are some other definitions we will use.

$$\begin{aligned} C_q &= \text{ the set of non-principal Dirichlet characters modulo } q, \\ C_q(a,b) &= \{\chi \in C_q : \chi(a) \ne \chi(b)\}, \\ \lambda(q) &= \text{ Carmichael's function: the largest order of an element of } F_q^*, \\ \lfloor x \rfloor &= \text{ the greatest integer which is } \leqslant x, \\ \{x\} &= x - \lfloor x \rfloor, \text{ the fractional part of } x, \\ e(z) &= e^{2\pi i z}. \end{aligned}$$

Constants implied by the Landau $O-$ and Vinogradov $\ll -$ symbols may depend on q, but not on any other variable.

We begin with a lemma showing the relationship between functions $\pi_{q,a}(x)$ and zeros of L-functions modulo q.

Lemma 1.1. *Let $q \geqslant 3$ and $a \in F_q^*$. Let $N_q(c)$ denote the number of incongruent solutions of the congruence $w^2 \equiv c \pmod q$, and let $\pi(x)$ be the number of primes $\leqslant x$. Then for $x \geqslant 2$,*

$$\phi(q)\pi_{q,a}(x) = \pi(x) - 2\Re\Bigg(\sum_{\chi \in C_q} \overline{\chi}(a) \sideset{}{^*}\sum_{\substack{L(\rho,\chi)=0 \\ \Im\rho \geqslant 0 \\ \Re\rho > 0}} f(\rho) \Bigg) - N_q(a)\frac{x^{1/2}}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x} \right),$$

where

$$f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho}\int_2^x \frac{t^\rho}{t\log^2 t}\, dt = \frac{x^\rho}{\rho\log x} + O\left(\frac{x^{\Re\rho}}{|\rho|^2 \log^2 x} \right),$$

zeros are counted with mutiplicity, and $\sum^*$ indicates that the summand is $\frac12 f(\rho)$ if $\Im\rho = 0$.

Lemma 1.1 is well-known, following from explicit formulas (e.g. [Da], chapters 19, 20). See also the proof of Lemma 1.1 of [FK1].

Corollary 1.2. *Let $\sigma > \frac{1}{2}$, $q \geqslant 3$ and $a, b \in F_q^*$. Then as $x \to \infty$,*

$$\phi(q)\left(\pi_{q,a}(x) - \pi_{q,b}(x)\right) = -2\Re\Bigg(\sum_{\chi \in C_q} (\overline{\chi}(a) - \overline{\chi}(b)) \sideset{}{^*}\sum_{\substack{L(\rho,\chi)=0 \\ \Im\rho \geqslant 0 \\ \Re\rho \geqslant \sigma}} f(\rho)\Bigg) + o\left(\frac{x^\sigma}{\log x}\right).$$

Corollary 1.2 is a very old result, and follows from Lemma 1.1 and bounds

$$\sum_{|\Im\rho| \geqslant x} \frac{x^\rho}{\rho} = o(x^{1/2}), \quad N(T,\chi) \ll T\log T, \quad N(T,\sigma,\chi) \ll_\sigma T^{1-\delta(\sigma)},$$

where $\delta(\sigma) > 0$ for $\sigma > 1/2$ and

$$N(T,\chi) = |\{\rho : |\Im\rho| \leqslant T, \Re\rho > 0\}|, \quad N(T,\sigma,\chi) = |\{\rho : |\Im\rho| \leqslant T, \Re\rho \geqslant \sigma\}|.$$

See for example a similar analysis for the approximation of $\pi(x)$ in [SP]. The first two estimates above can be found in Davenport ([Da], Ch. 19, 20) and an example of the third can be found in Montgomery (e.g. [Mo], Theorem 12.1). The upper bound on $N(T,\sigma,\chi)$ implies that

$$\sum_{\chi \in C_q} \sum_{\substack{L(\rho,\chi)=0 \\ \Re\rho \geqslant \sigma \\ \Im\rho > T}} \frac{1}{|\rho|} \ll_\sigma T^{-\delta(\sigma)}. \tag{1.1}$$

In applying Corollary 1.2, frequently we approximate $f(\rho)$ by $x^\rho/(\rho\log x)$ with a total error of at most

$$O\Bigg(\frac{x^\sigma}{\log^2 x} \sum_{\chi \in C_q(a,b)} \sum_{\substack{L(\rho,\chi)=0 \\ \Re\rho \geqslant 1/2}} \frac{1}{|\rho|^2}\Bigg) = O\left(\frac{x^\sigma}{\log^2 x}\right).$$

Therefore we have the following.

Corollary 1.3. *Let $q \geqslant 3$, $a, b \in F_q^*$, $\sigma > 1/2$ and suppose for $\chi \in C_q(a,b)$, the zeros of $L(s,\chi)$ have real part $\leqslant \sigma$. Then, as $u \to \infty$,*

$$\frac{u\phi(q)}{2e^{\sigma u}}\left(\pi_{q,a}(e^u) - \pi_{q,b}(e^u)\right) = \sum_{\chi \in C_q(a,b)} \sideset{}{^*}\sum_{\substack{L(\sigma+it,\chi)=0 \\ t \geqslant 0}} \frac{\nu(b) - \nu(a)}{\sqrt{t^2+\sigma^2}} + o(1),$$

where $\nu(n) = \sin(tu - \operatorname{Arg}\chi(n) + \tan^{-1}(\sigma/t))$. Here we adopt the convention that $\tan^{-1}(\sigma/t) = \pi/2$ if $t = 0$.

An inequality which is useful when t is large is

$$|\sin(v + \tan^{-1}(\sigma/t)) - \sin(v)| \leqslant \tan^{-1}(\sigma/t) \leqslant \sigma/t. \tag{1.2}$$

Questions concerning the signs of the differences $\pi_{q,a}(x) - \pi_{q,b}(x)$ therefore boil down to questions about the trigonometric sums occurring in Lemma 1.1 and Corollaries 1.2,1.3. As opposed to [FK1], a *barrier* in this paper refers to the existence of a system of trigonometric sums of this type with certain properties, and has nothing directly to do with prime counting functions. All of our results on the existence or non-existence of particular types of barriers have consequences for the distribution of functions $\pi_{q,a}(x)$, but it is important to separate the two.

Suppose for each $\chi \in C_q$, $B(\chi)$ is a sequence of complex numbers with non-negative imaginary part (possibly empty, duplicates allowed), and denote by $\mathcal{B}$ the system of $B(\chi)$ for $\chi \in C_q$. Let $n(\rho,\chi)$ be the number of occurrences of the number ρ in $B(\chi)$. If ρ is real, we suppose that $n(\rho,\chi) = n(\rho,\overline{\chi})$. The sets $B(\chi)$ will play the role of hypothetical zeros of the L-function $L(s,\chi)$. Define

$$R^+(\mathcal{B}) = \sup\{\Re\rho : \rho \in \mathcal{B}\}, \quad R^-(\mathcal{B}) = \inf\{\Re\rho : \rho \in \mathcal{B}\}.$$

We shall suppose throughout that

$$\frac{1}{2} < R^-(\mathcal{B}) \leqslant R^+(\mathcal{B}) \leqslant 1 \tag{1.3}$$

and also, in accordance with (1.1), that

$$\sum_{\chi \in C_q} \sum_{\rho \in B(\chi)} \frac{n(\rho,\chi)}{|\rho|} < \infty. \tag{1.4}$$

In accordance with Lemma 1.1, define

$$P_{q,a}(x;\mathcal{B}) = -\frac{2}{\phi(q)} \Re\Bigg(\sum_{\chi \in C_q} \overline{\chi}(a) \sum_{\rho \in B(\chi)}{}^{*} n(\rho,\chi) f(\rho) \Bigg) + \frac{\pi(x)}{\phi(q)} \tag{1.5}$$

and

$$D_{q,a,b}(x;\mathcal{B}) = P_{q,a}(x;\mathcal{B}) - P_{q,b}(x;\mathcal{B}).$$

where as before $\sum^*$ means the inner summand is $\frac{n(\rho,\chi)}{2} f(\rho)$ when ρ is real. We say that two functions $F_1, F_2 : [0,\infty) \to \mathbb{R}$ are β-similar if $|F_1(x) - F_2(x)| = o(x^\beta/\log x)$ as $x \to \infty$. This is related to the conclusions in Lemma 1.1 and Corollary 1.2. For indexed sets of functions $\mathcal{F} = \{F_i\}, \mathcal{G} = \{G_i\}$, we say that $\mathcal{F}$ and $\mathcal{G}$ are β-similar if F_i and G_i are β-similar for each i. With q and $\mathcal{B}$ fixed, let $\mathcal{P}_q$ be the list of functions $P_{q,a}(x;\mathcal{B})$. For a system of functions $\mathcal{F}$, also indexed by $a \in F_q^*$, suppose $\mathcal{I}(\mathcal{F})$ is a statement concerning the magnitudes of functions $F_{q,a}(x)$. An example is

For sufficiently large x, at least one $F_{q,a}(x) < F_{q,1}(x) \quad (a \in F_q^* \backslash \{1\})$.

For a system $\mathcal{B}$, let $\beta = R^-(\mathcal{B})$. We say that $\mathcal{B}$ is a *barrier for* $\mathcal{I}$ if, for every $\mathcal{F}$ which is β-similar to $\mathcal{P}_q$, $\mathcal{I}(\mathcal{F})$ is false.

To relate this to the prime race problem, let Π_q be the list of functions $\pi_{q,a}(x)$, indexed by $a \in F_q^*$. Let $z_{\mathcal{B}}$ denote the condition that for each $\chi \in C_q$ and $\rho \in B(\chi)$, $L(s,\chi)$ has a zero of multiplicity $n(\rho,\chi)$ at $s=\rho$, and all other zeros of $L(s,\chi)$ in the upper half plane have real part less than $R^-(\mathcal{B})$. By Lemma 1.1, if $z_{\mathcal{B}}$ then Π_q is β-similar to $\mathcal{P}_q$, thus we have the following.

Lemma 1.4. *If $\mathcal{B}$ is a barrier for $\mathcal{I}$ and condition $z_{\mathcal{B}}$ holds, then $\mathcal{I}(\Pi_q)$ is false.*

If each sequence $B(\chi)$ is finite, we call $\mathcal{B}$ a *finite barrier* for $\mathcal{I}$ and denote by $|\mathcal{B}|$ the sum of the number of elements of each sequence $B(\chi)$, counted according to multiplicity. We say that $|\mathcal{B}|$ is the size of the barrier $\mathcal{B}$. Of primary interest is to construct barriers for $\mathcal{I}$ where the imaginary parts of the points in each $B(\chi)$ are all $\geqslant \tau$ for an arbitrarily large τ. It may occur that $|\mathcal{B}|$ remains bounded as $\tau \to \infty$, in which case we say that $\mathcal{I}$ possesses a *bounded barrier* (which is actually a sequence of barriers). Later we will demonstrate the non-existence of bounded barriers for certain statements $\mathcal{I}$. There is one more type of barrier which we will work with, the *extremal barrier*, which will be defined in section 4. Finally, we remark that in general we can choose $R^-(\mathcal{B})$ and $R^+(\mathcal{B})$ arbitrarily as long as $1/2 < R^-(\mathcal{B})$, $R^+(\mathcal{B}) < 1$.

An important feature of the sums $D_{q,a,b}(x;\mathcal{B})$ is that the "dominant parts" are often almost periodic functions. To be specific, let

$$(1.6)\quad g(\rho) = g(\rho;a,b) = \sum_{\chi\in C_q} n(\rho,\chi)(\chi(a)-\chi(b)), \quad \beta(a,b) = \sup\{\Re\rho : g(\rho)\neq 0\}.$$

Also let

$$(1.7)\quad z(\chi;a,b) = \{\rho\in B(\chi) : g(\rho)\neq 0, \Re\rho = \beta(a,b)\}, \quad z(a,b) = \bigcup_{\chi\in C_q} z(\chi;a,b).$$

In essence, the numbers in $z(a,b)$ are the ones which produce the dominant terms in $D_{q,a,b}(x;\mathcal{B})$, provided $z(a,b)$ is non-empty. Writing $\beta = \beta(a,b)$ for brevity, we have

(1.8)

$$\begin{aligned} D_{q,a,b}(x;\mathcal{B}) &= \frac{2x^\beta}{\phi(q)\log x} M_{q,a,b}(x;\mathcal{B}) + E_{q,a,b}(x;\mathcal{B}), \\ M_{q,a,b}(x;\mathcal{B}) &:= -\Re\Bigg(\sum_{\chi\in C_q} (\overline{\chi}(a)-\overline{\chi}(b)) \sideset{}{^*}\sum_{\beta+i\gamma\in z(\chi;a,b)} n(\beta+i\gamma,\chi)\frac{x^{i\gamma}}{\beta+i\gamma}\Bigg) \\ &= -\Re\Bigg(\sideset{}{^*}\sum_{\beta+i\gamma\in z(a,b)} \overline{g(\beta+i\gamma)}\frac{x^{i\gamma}}{\beta+i\gamma}\Bigg). \end{aligned}$$

Using Lemma 1.1 and (1.4), we have

$$(1.9)\quad \begin{aligned} |E_{q,a,b}(x;\mathcal{B})| &\ll \sum_{\chi\in C_q}\Bigg[\sideset{}{^*}\sum_{\rho\in B(\chi)} \frac{x^{\Re\rho}}{|\rho|^2\log^2 x} + \sideset{}{^*}\sum_{\substack{\rho\in B(\chi)\\ \rho\notin z(\chi;a,b)}} \frac{x^{\Re\rho}}{|\rho|\log x}\Bigg] \\ &\ll \frac{x^\beta}{\log^2 x} + o\left(\frac{x^\beta}{\log x}\right) \\ &= o\left(\frac{x^\beta}{\log x}\right) \quad (x\to\infty). \end{aligned}$$

A function f is said to be *almost periodic* with respect to a norm $\|\cdot\|$ if for and $\varepsilon > 0$, there is an $L > 0$, so that any real interval of length L contains a number τ so that

$$\|f(u+\tau) - f(u)\| \leqslant \varepsilon.$$

It follows from (1.4) and Theorems 8 and 12 of §1 of Chapter 1 in [Be] that each sum $M_{q,a,b}(e^u; \mathcal{B})$ is a uniformly continuous almost periodic function in the sense of Bohr; that is, almost periodic with respect to the supremum norm. If one takes $B(\chi)$ to be the set of zeros ρ of $L(s,\chi)$ with $\Re\rho = \beta$ and $\Im\rho \geqslant 0$ (for $\chi \in C_q(a,b)$), then $M_{q,a,b}(x; \mathcal{B})$ is precisely the double sum appearing in the conclusion of Corollary 1.3 with $\sigma = \beta$. Thus this double sum is also a uniformly continuous almost periodic function in the sense of Bohr. For a uniformly continuous almost periodic function f, define

$$\|f\|_2 = \lim_{U\to\infty} \left(\frac{1}{U} \int_0^U |f^2(u)|du \right)^{1/2}$$

(the limit exists by Theorem 2 of §3 of Chapter 1 in [Be]). Next, if $f_1, \ldots, f_k$ are almost periodic with respect to a norm $\|\cdot\|_A$, then the vector-valued function

$$f(u) = (f_1(u), \ldots, f_k(u))$$

is almost periodic with respect to the norm

$$\|f\|_B := \max_{1\leqslant j\leqslant k} \|f_j\|_A.$$

If, for some $\chi \in C_q$, $\chi(a) \neq \chi(b)$ and all non-trivial zeros of $L(s,\chi)$ have real part 1/2 (the Extended Riemann Hypothesis for χ), the inner sum in Corollary 1.2 (with $\sigma = 1/2$) is not uniformly convergent (in fact, it has infinitely many jump discontinuities), but it is still almost periodic in the sense of Stepanov ([Be], chapter 2). That is, it is almost periodic with respect to the norm

$$\|g\|_{S^2} := \max_{x\in\mathbb{R}} \left(\int_x^{x+1} |g(y)|^2\, dy \right)^{1/2}.$$

The proof of this is implicit in [K2]; another proof and generalization can be found in [KR]. We note that if a function is almost periodic in the Bohr sense, it is also almost periodic in the Stepanov sense, since $\|g\|_{S^2} \leqslant \|g\|_\infty$. Any function g, almost period function in the Stepanov sense, has the property that if u is a continuity point of g, then for every $\varepsilon > 0$ there is an unbounded set of v so that $|g(v) - g(u)| \leqslant \varepsilon$.

Remark 1.1. When each function in a set $\mathcal{F}$ is almost periodic in the Stepanov sense, to prove that some set of (strict) inequalities among a set of functions $\mathcal{F}$ occurs for an an bounded set of u, it suffices to prove that the set of inequalities occur for a single u which is a continuity point of each function. We can in fact make a stronger conclusion: for some L and $\delta > 0$, on any interval of length L, the measure of the set of u for which the set of inequalities occur is $\geqslant \delta$.

As a consequence, setting $u = \log x$, we conclude that the set of inequalities occurs on a set of x of positive lower asymptotic density.

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2. Signs and comparison of trigonometric polynomials

First, we formulate some simple properties of trigonometric polynomials. In particular, we prove that a real n-term trigonometric polynomial with a zero constant term must be nonnegative on a large set. By $\mu(E)$, where $E \subset \mathbf{R}$, we denote the Lebesgue measure of E.

Lemma 2.1. *Let P be a real trigonometric polynomial $P(u) = \sum_{k=1}^{n} c_k \sin(t_k u + \alpha_k)$ $(c_k \in \mathbf{R}, t_k \in \mathbf{R}, t_k \neq 0, t_k \neq t_l (k \neq l))$, $E_+ = \{u : P(u) \geqslant 0\}$. Then*
1) $\int_0^U P(u)du = o(U) \quad (U \to \infty)$;
2) $\|P\|_2 := \lim_{U\to\infty} \left(\frac{1}{U}\int_0^U (P(u))^2 du\right)^{1/2} = \left(\frac{1}{2}\sum_{k=1}^n c_k^2\right)^{1/2}$;
3) $\|P\|_\infty \geqslant \|P\|_2 \geqslant \sqrt{\frac{1}{2n}}\sum_{k=1}^n |c_k|$, where $\|P\|_\infty := \sup_u |P(u)|$;
4) $\sup_u P(u) \geqslant \max_k |c_k|/2 \geqslant \|P\|_\infty/2n$;
5) $\mu(E_+ \cap [0,U])/U \geqslant \frac{1}{4n} + o(1) \quad (U \to \infty)$.

Proof. We have

$$\int_0^U P(u)du = \sum_{k=1}^n \frac{c_k}{t_k}(\cos\alpha_k - \cos(t_k U + \alpha_k)). \tag{2.1}$$

The right-hand side of (2.1) is bounded for $u \in \mathbf{R}$, and 1) follows.

Further,

$$\begin{aligned}\int_0^U (P(u))^2 du &= \sum_{k,l=1}^n c_k c_l \int_0^U \sin(t_k u + \alpha_k)\sin(t_l u + \alpha_l)du \\ &= \sum_{k=1}^n \int_0^U \left(\frac{c_k^2}{2} + \frac{c_k^2}{2}\cos((2t_k u + 2\alpha_k))\right) \\ &\quad + \sum_{k\neq l}^n c_k c_l \int_0^U \sin(t_k u + \alpha_k)\sin(t_l u + \alpha_l)du,\end{aligned}$$

and, by 1),

$$\int_0^U (P(u))^2 du = U\sum_{k=1}^n \frac{c_k^2}{2} + o(U) \quad (U \to \infty).$$

The first part in 3) follows from the inequality

$$\int_0^U (P(u))^2 du \leqslant \int_0^U \|P\|_\infty^2.$$

The second part follows from 2) and the Cauchy-Schwarz inequality.

To prove 4), we take l so that $|c_l| = \max_k |c_k|$. Without loss of generality, $c_l \geqslant 0$. Denote $a = \sup_u P(u)$. For $U > 0$ we have

(2.2)
$$\begin{aligned} 0 &\geqslant \int_0^U (P(u)-a)(\sin(t_l u+\alpha_l)+1)du \\ &= -aU - a\int_0^U \sin(t_l u+\alpha_l)du + \int_0^U P(u)du + \int_0^U P(u)\sin(t_l u+\alpha_l)du \\ &= -aU + \int_0^U P(u)\sin(t_l u+\alpha_l)du + o(U) \quad (U\to\infty), \end{aligned}$$

by 1). Further, using again 1) and 2), we get

(2.3)
$$\begin{aligned} \int_0^U P(u)\sin(t_l u+\alpha_l)du &= c_l\int_0^U (\sin(t_l u+\alpha_l))^2du \\ &\quad + \sum_{k\neq l}\int_0^U c_k\sin(t_k u+\alpha_k)\sin(t_l u+\alpha_l)du \\ &= \frac{c_l}{2}U + o(U) \quad (U\to\infty), \end{aligned}$$

and 4) follows from (2.2) and (2.3).

Denote $E_- = \{u:\ P(u)\leqslant 0\}$. By 1),

$$\int_{E_+\cap[0,U]} P(u)du = -\int_{E_-\cap[0,U]} |P(u)|du + o(U) \quad (U\to\infty).$$

Therefore,

$$\int_{E_+\cap[0,U]} P(u)du = \frac{1}{2}\int_0^U |P(u)|du + o(U) \quad (U\to\infty). \tag{2.4}$$

On the other hand, taking again $|c_l| = \max_k |c_k|$, we have, by (2.3),

$$\int_0^U |P(u)|du \geqslant \left|\int_0^U P(u)\sin(t_l u+\alpha_l)du\right| = \frac{|c_l|}{2}U + o(U) \quad (U\to\infty).$$

The equality (2.4) implies

$$\int_{E_+\cap[0,U]} P(u)du \geqslant \frac{|c_l|}{4}U + o(U) = \frac{\max_k |c_k|}{4}U + o(U) \quad (U\to\infty). \tag{2.5}$$

Note that

$$\max_k |c_k| \geqslant \frac{1}{n}\sum_{k=1}^n |c_k| \geqslant \frac{1}{n}\|P\|_\infty.$$

Therefore,

$$\int_{E_+\cap[0,U]} P(u)du \leqslant \|P\|_\infty \mu(E_+\cap[0,U]) \leqslant n\max_k |c_k|\mu(E_+\cap[0,U]). \tag{2.6}$$

We exclude a trivial case when P is not identically zero. Then combination of (2.5) and (2.6) proves 5) and thus completes the proof of Lemma. $\square$

Theorem 2.2. [N] *Let P be an exponential polynomial $P(u)=\sum_{k=1}^n c_k e^{it_k u}$ ($c_k\in \mathbf{C}, t_k\in\mathbf{R}$), $U>0$, $E\subset[0,U]$ of positive Lebesgue measure: Then*

$$\max_{u\in[0,U]}|P(u)|\le\left\{\frac{CU}{\mu(E)}\right\}^{n-1}\sup_{u\in E}|P(u)|,$$

where C is an absolute constant.

Corollary 2.3. *Let P be a real trigonometric polynomial $P(u)=\sum_{k=1}^n c_k\sin(t_k u+\alpha_k)$ ($c_k\in\mathbf{R}, t_k\in\mathbf{R}, t_k\ne 0, t_k\ne t_l (k\ne l)$), $0<\gamma<1$, $S=\sum_{k=1}^n|c_k|$, $\varepsilon=\frac{1}{2\sqrt{n}}(C/\gamma)^{1-2n}$, where C is the constant from Theorem 2.2, $E=\{u:\ |P(u)|<\varepsilon S\}$. Then for sufficiently large U*

$$\mu(E\cap[0,U])/U<\gamma.$$

Proof. Using Lemma 2.1, we get for sufficiently large U

$$\max_{u\in[0,U]}|P(u)|\geqslant\frac{S}{\sqrt{3n}}.$$

Suppose that

$$\mu(E\cap[0,U])/U\geqslant\gamma. \tag{2.7}$$

Then, writing P in an exponential form with $2n$ terms, we get from Theorem 2.2,

$$\frac{S}{\sqrt{3n}}\le(C/\gamma)^{2n-1}\sup_{u\in E}|P(u)|\le(C/\gamma)^{2n-1}\varepsilon S,$$

and, by the definition of ε, $S=0$, but in this case $E=\emptyset$. Thus, the supposition (2.7) cannot hold, and Corollary is proved. □

Lemma 2.4. *For any positive integer n there exists such $\varepsilon_1=\varepsilon_1(n)>0$ that if P,Q are real trigonometric polynomials,*

$$P(u)=\sum_{k=1}^n a_k\cos(t_k u+\alpha_k),\quad Q(u)=\sum_{k=1}^n b_k\sin(t_k u+\beta_k),$$

$$t_k\ne 0,\ t_k\ne t_l(k\ne l),\quad |\alpha_k|\leqslant\varepsilon_1,\ |\beta_k|\leqslant\varepsilon_1\quad(k=1,\dots,n),$$

then there exists a real number u such that

$$P(u)\geqslant\varepsilon_1\sum_{k=1}^n|a_k|,\quad Q(u)\geqslant\varepsilon_1\sum_{k=1}^n|b_k|.$$

Proof. Take $\gamma=1/(10n)$. We will prove the lemma for

$$\varepsilon_1=\varepsilon/2, \tag{2.8}$$

where ε is chosen in accordance with Corollary 2.3. Denote

$$\tilde{P}(u) = \sum_{k=1}^{n} a_k \cos(t_k u), \quad S_1 = \sum_{k=1}^{n} |a_k|.$$

Let $E = \{u : \ \tilde{P}(u) \geqslant 0\}$, $E_1 = \{u : \ |\tilde{P}(u)| < 2\varepsilon_1 S_1\}$. Thus,

$$\forall u \in (E \setminus E_1) \quad \tilde{P}(u) \geqslant 2\varepsilon_1 S_1. \tag{2.9}$$

Take a sufficiently large U. By Lemma 2.1, we have

$$\mu(E \cap [0, U])/U \geqslant \frac{1}{5n}. \tag{2.10}$$

Also, by Corollary 2.3 and (2.8),

$$\mu(E_1 \cap [0, U])/U < \frac{1}{10n}. \tag{2.11}$$

Let

$$\tilde{Q}(u) = \sum_{k=1}^{n} b_k \sin(t_k u), \quad S_2 = \sum_{k=1}^{n} |b_k|, \quad E_2 = \{u : \ |\tilde{Q}(u)| < 2\varepsilon_1 S_2\}.$$

By Corollary 2.3 and (2.8),

$$\mu(E_2 \cap [0, U])/U < \frac{1}{10n}. \tag{2.12}$$

The inequalities (2.10)—(2.12) show that the set $E' = E \setminus E_1 \setminus E_2$ is nonempty. Using evenness of $\tilde{P}$ and oddness of $\tilde{Q}$ we obtain that for $u_1 \in E'$ either $u = u_1$ or $u = -u_1$ satisfies the inequalities

$$\tilde{P}(u) \geqslant 2\varepsilon_1 S_1, \quad \tilde{Q}(u) \geqslant 2\varepsilon_1 S_2.$$

Taking into account, that, by the restrictions on α_k and β_k, we have

$$|P(u) - \tilde{P}(u)| \leqslant \varepsilon_1 S_1, \quad |Q(u) - \tilde{Q}(u)| \leqslant \varepsilon_1 S_2,$$

we get

$$P(u) \geqslant \varepsilon_1 S_1, \quad Q(u) \geqslant \varepsilon_1 S_2,$$

as required. □

The following lemma is closed to Lemma 1 from [FFK].

Lemma 2.5. *Let n be a positive integer, $0 < \alpha < 1$,*

$$\varepsilon = \varepsilon(n, \alpha) = 6(\alpha/6)^{2^{n-1}},$$

$s_1 > \cdots > s_n > 0$. Then there exists a real number u such that $\varepsilon \leqslant \{us_k\} \leqslant \alpha$ for each $k \in \{1, \ldots, n\}$.

Proof. We use induction on n. For $n = 1$ we have $\varepsilon = \alpha$ and the statement is trivial. Suppose that $n > 1$ and the lemma holds for $n - 1$. We use the induction supposition for $\alpha' = \alpha^2/6$ instead of α and for $\{s_2, \ldots, s_n\}$. Observe that $\varepsilon = \varepsilon(n, \alpha) = \varepsilon(n - 1, \alpha')$. There exists a real number u' such that $\varepsilon \leqslant \{u's_k\} \leqslant \alpha'$ for each $k \in \{2, \ldots, n\}$. By Dirichlet's box principle, there exists a positive integer l satisfying $l \leqslant 3/\alpha$ and $\|lu's_1\| \leqslant \alpha/3$. Take $u = lu' + \alpha/(2s_1)$. We have

$$\alpha/6 \leqslant \{us_1\} \leqslant 5\alpha/6$$

and for $k \in \{2, \ldots, n\}$

$$\{us_k\} \leqslant l\{u's_k\} + \alpha s_k/(2s_1) \leqslant l\alpha' + \alpha/2 \leqslant \alpha,$$

$$\{us_k\} > \varepsilon,$$

as required.

Lemma 2.6. *Let n be a positive integer,*

$$\varepsilon_2 = \varepsilon_2(n) = 13^{-2^{n-1}},$$

t_k be positive numbers, $|\beta_k| \leqslant \varepsilon_2$ for $k = 1, \ldots, n$. Then there exists a real number u such that $\sin(t_k u + \beta_k) < -\varepsilon_2$ for each $k \in \{1, \ldots, n\}$.

Proof. Take $\alpha = 6/13$ and $s_k = t_k/(2\pi)$ for $k = 1, \ldots, n$. By Lemma 2.5, there is u' such that $\varepsilon_2 \leqslant \{u's_k\} \leqslant \alpha$ for each $k \in \{1, \ldots, n\}$. It is easy to check that $u = -2\pi u'$ satisfies Lemma 2.6.

Lemma 2.7. *For any positive integer n there exists such $\varepsilon_3 = \varepsilon_3(n) > 0$ that for any real $\gamma > 0$ and real trigonometric polynomials*

$$P(u) = \sum_{k=1}^{n} a_k \cos(t_k u),$$

$$Q(u) = \sum_{k=1}^{n} b_k \sin(t_k u),$$

$$R(u) = \sum_{k=1}^{n} c_k \sin(t_k u),$$

$$t_k \neq 0, t_k \neq t_l (k \neq l), \quad b_k \geqslant |a_k| + c_k, \quad c_k \geqslant 0, \quad (k = 1, \ldots, n),$$

$$\sum_{k=1}^{n} |a_k| > \gamma \sum_{k=1}^{n} b_k,$$

there exists a real number u such that

$$Q(u) > \max(|P(u)|, R(u)) + \varepsilon_3 \gamma^2 \sum_{k=1}^{n} b_k.$$

The basic idea of the proof is the inequality $\|Q\|_2^2 \geqslant \|P\|_2^2 + \|R\|_2^2$ following from Lemma 2.1. This inequality shows that there is a real u such that

$$Q^2(u) \geqslant P^2(u) + R^2(u) \geqslant \max(P^2(u), R^2(u)). \tag{2.13}$$

To strengthen (2.13), one can use the following possibilities:
1) to estimate $\|Q\|_2^2 - \|P\|_2^2 - \|R\|_2^2$ from below;
2) to estimate $\min(P^2(u), R^2(u))$ from below and thus to strengthen the inequality $P^2(u) + R^2(u) \geqslant \max(P^2(u), R^2(u))$;
3) to show that $Q^2 - P^2 - R^2$ is not close to a constant and thus has a big positive value at some point.
It depends on the situation which of these arguments can work. First we will prove a lemma using arguments 1) and 2).

Lemma 2.8. *Under the suppositions of Lemma 2.7, there exists $\varepsilon_4 = \varepsilon_4(n) > 0$ and a real number u such that*

$$Q^2(u) > \max(P^2(u), R^2(u)) + \max_k(\min((b_k^2 - a_k^2 - \gamma b_k^2)/2, \varepsilon_4 \gamma^2 b_k^2)).$$

Proof. Take any $k_0 \in \{1, \dots, n\}$. If $c_{k_0} < \gamma b_{k_0}$, then

$$\|Q\|_2^2 - \|P\|_2^2 - \|R\|_2^2 = \frac{1}{2} \sum_{k=1}^{n} (b_k^2 - a_k^2 - c_k^2) \geqslant \frac{1}{2}(b_{k_0}^2 - a_{k_0}^2 - c_{k_0}^2) > \frac{1}{2}(b_{k_0}^2 - a_{k_0}^2 - \gamma^2 b_{k_0}^2).$$

Therefore, there is u such that

$$Q^2(u) - P^2(u) - R^2(u) > \frac{1}{2}(b_{k_0}^2 - a_{k_0}^2 - \gamma^2 b_{k_0}^2). \tag{2.14}$$

Now let us consider the case $c_{k_0} \geqslant \gamma b_{k_0}$, Let ε be the number from Lemma 2.3, corresponding to $\gamma = 1/3$,

$$E_1 = \{u : \ |R(u)| < \varepsilon \gamma b_{k_0}\}.$$

By Lemma 2.3, for sufficiently large U

$$\mu(E_1 \cap [0, U])/U < 1/3. \tag{2.15}$$

Also, let

$$E_2 = \{u : \ |P(u)| < \varepsilon\gamma b_{k_0}\}.$$

Taking into account the supposition of Lemma 2.7 for $\sum_{k=1}^n |a_k|$, we get from Lemma 2.3

$$\mu(E_2 \cap [0,U])/U < 1/3. \tag{2.16}$$

Set $E_3 = [0,U] \setminus E_1 \setminus E_2$. By (2.15) and (2.16), we have

$$\mu(E_3) > U/3. \tag{2.17}$$

Also, from the definitions of E_1 and E_2 we find that for every $u \in E_3$

$$\min(|P(u)|, |R(u)|) \geqslant \varepsilon\gamma b_{k_0}. \tag{2.18}$$

Using Lemma 2.1, (2.17) and (2.18), we get

$$\begin{aligned}\int_0^U Q^2(u)du &\geqslant \int_0^U (P^2(u) + R^2(u))du + o(U) \\ &= \int_0^U \max(P^2(u), R^2(u))du + \int_0^U \min(P^2(u), R^2(u))du + o(U) \\ &\geqslant \int_0^U \max(P^2(u), R^2(u))du + \int_{E_3} (\varepsilon\gamma b_{k_0})^2 du + o(U) \\ &\geqslant \int_0^U \max(P^2(u), R^2(u))du + (\varepsilon^2\gamma^2 b_{k_0}^2/3)U + o(U) \quad (U \to \infty).\end{aligned}$$

Hence, there exists $u \in [0,U]$ such that

$$Q^2(u) - P^2(u) - R^2(u) > \varepsilon_4\gamma^2 b_{k_0}^2, \quad \varepsilon_4 = \varepsilon^2/4. \tag{2.19}$$

So, for every k_0 one of the inequalities (2.14), (2.19) holds. This proves Lemma 2.8.

Proof of Lemma 2.7. Without loss of generality, we can consider $t_k > 0$ for $k = 1, \dots, n$ and $\gamma \leqslant 1/2$. Let $\beta_1 = \frac{1}{n}$, $\beta_j = \frac{\beta_{j-1}^2}{16}$ for $j = 2, \dots, n+1$,

$$S = \sum_{k=1}^n b_k.$$

Choose the numbers $k_1, k_2, \dots$ so that

$$b_{k_1} \geqslant \beta_1 S, \quad t_{k_j} > t_{k_{j-1}}, \ b_{k_j} \geqslant \beta_j S \ (j > 1).$$

Note that k_1 can be always found because

$$\max_k b_k \geqslant \frac{1}{n}\sum_k b_k = \beta_1 S.$$

We terminate our construction when for some l we cannot define a following number k_{l+1}, that is

$$\forall t_k > t_{k_l},\ b_k < \beta_{l+1} S. \tag{2.20}$$

If $|a_{k_l}| < b_{k_l}/2$, then, by Lemma 2.8, there exists a real number u such that

$$Q^2(u) > \max(P^2(u), R^2(u)) + \min((b_{k_l}^2 - a_{k_l}^2 - \gamma b_{k_l}^2)/2, \varepsilon_4 \gamma^2 b_{k_l}^2).$$

Further,

$$b_{k_l}^2 - a_{k_l}^2 - \gamma b_{k_l}^2 > \frac{1}{4} b_{k_l}^2,$$
$$b_{k_l} \geqslant \beta_n S.$$

Therefore,

$$Q^2(u) > \max(P^2(u), R^2(u)) + \min(1/8, \varepsilon_4\gamma^2)\beta_n^2 S^2. \tag{2.21}$$

Now we have to consider the case

$$|a_{k_l}| \geqslant b_{k_l}/2. \tag{2.22}$$

Define the even trigonometric polynomial $W(u) = Q^2(u) - P^2(u) - R^2(u)$ and estimate the coefficient A of $\cos(Tu)$ in W, $T = 2t_{k_l}$. We have

$$\begin{aligned} A = -(a_{k_l}^2 + b_{k_l}^2 - c_{k_l}^2)/2 &- \sum_{\substack{t_k + t_{k'} = T,\\ t_{k_l} < t_k < 2t_{k_l}}} (a_k a_{k'} + b_k b_{k'} - c_k c_{k'}) \\ &- \sum_{\substack{t_k - t_{k'} = T,\\ t_k > 2t_{k_l}}} (a_k a_{k'} + c_k c_{k'} - b_k b_{k'}). \end{aligned} \tag{2.23}$$

By (2.22),

$$(a_{k_l}^2 + b_{k_l}^2 - c_{k_l}^2)/2 \geqslant (a_{k_l}^2 + b_{k_l}^2 - (b_{k_l} - |a_{k_l}|)^2)/2 = |a_{k_l}| b_{k_l} \geqslant b_{k_l}^2/2 \geqslant \beta_l^2 S^2/2. \tag{2.24}$$

For $t_k > t_{k_l}$ and arbitrary k' we have, by (2.20),

$$|a_k a_{k'}| + b_k b_{k'} + c_k c_{k'} \leqslant 2 b_k b_{k'} \leqslant 2\beta_{l+1} b_{k'} S = \beta_l^2 b_{k'} S/8.$$

Therefore,

$$\begin{aligned} \sum_{\substack{t_k + t_{k'} = T,\\ t_{k_l} < t_k < 2t_{k_l}}} (a_k a_{k'} + b_k b_{k'} - c_k c_{k'}) &+ \sum_{\substack{t_k - t_{k'} = T,\\ t_k > 2t_{k_l}}} (a_k a_{k'} + c_k c_{k'} - b_k b_{k'}) \\ &\geqslant -2 \sum_{k'} \beta_l^2 b_{k'} S/8 \geqslant -\beta_l^2 S^2/4. \end{aligned} \tag{2.25}$$

Substituting (2.24) and (2.25) into (2.23), we get

$$A \leqslant -\beta_l^2 S^2/4 \leqslant -\beta_n^2 S^2/4.$$

By Lemma 2.1, taking into account that W has a nonnegative constant term we obtain

$$\sup_u W(u) \geqslant \beta_n^2 S^2/8.$$

Thus, in the case (2.22), there exists u such that

$$Q^2(u) - P^2(u) - R^2(u) > \beta_n^2 S^2/9.$$

In the opposite case we had the inequality (2.21). So, for some $\varepsilon = \varepsilon(n)$ we always can find a real number u_1 such that

$$Q^2(u_1) > \max(P^2(u_1), R^2(u_1)) + \varepsilon\gamma^2 S^2.$$

Let $x = |Q(u_1)|$, $y = \max(|P(u_1)|, |R(u_1)|) < x$. Using the inequality $x - y > (x^2 - y^2)/(2x)$ we get

$$\begin{aligned} |Q(u_1)| &> \max(|P(u_1)|, |R(u_1)|) + \varepsilon\gamma^2 S^2/(2|Q(u_1)|) \\ &\geqslant \max(|P(u_1)|, |R(u_1)|) + \varepsilon\gamma^2 S/2, \end{aligned}$$

and either $u = u_1$ or $u = -u_1$ satisfies the required inequalities with $\varepsilon_4 = \varepsilon/2$. Lemma 2.7 is proved.

3. Player 1 leading and trailing

For short, we abbreviate the phrase "For arbitrarily large x" by "FAL x". In this section we address questions of whether or not

$$\text{FAL } x,\ \pi_{q,1}(x) < \pi_{q,a}(x) \quad (\forall a \in D), \tag{3.1}$$

$$\text{FAL } x,\ \pi_{q,1}(x) > \pi_{q,a}(x) \quad (\forall a \in D), \tag{3.2}$$

for various subsets D of $F_q^*\backslash\{1\}$. The residue 1 mod q is special because it is the identity in F_q^*, and this allows one to prove results about comparing $\pi_{q,1}(x)$ to $\pi_{q,a}(x)$ which would be difficult otherwise. For example, in the cases $q = 3, 4, 6$, $D = \{q-1\}$, Littlewood [Li] proved each of (3.1) and (3.2). Knapowski and Turán [KT1] proved that under the assumption that for each $\chi \in C_q$, $L(s,\chi)$ has no zeros on the real segment $(0,1)$ (known as Haselgrove's condition for q) that the difference $\pi_{q,1}(x) - \pi_{q,a}(x)$ changes sign infinitely often. Assuming the real parts of the nontrivial zeros of $L(s,\chi)$ are all 1/2 for $\chi \in C_q$, Kaczorowski [K2] proved that

$$\text{FAL } x,\ \pi_{q,1}(x) < \pi_{q,a}(x) \quad (\forall a \in F_q^*\backslash\{1\}), \tag{3.3}$$

$$\text{FAL } x,\ \pi_{q,1}(x) > \pi_{q,a}(x) \quad (\forall a \in F_q^*\backslash\{1\}). \tag{3.4}$$

In fact his proof gives a little bit more: if $D \subset F_q^*, 1 \notin D$, and all nontrivial zeros of $L(s,\chi)$ $(\chi \in \cup_{a\in D} C_q(a,1))$ have real part 1/2, then each of the inequalities (3.1) and (3.2) is true.

The statements pertaining to barriers which correspond to (3.1)–(3.4) are

$$\text{FAL } x,\ F_{q,1}(x) < F_{q,a}(x) \quad (\forall a \in D), \tag{3.1'}$$

$$\text{FAL } x,\ F_{q,1}(x) > F_{q,a}(x) \quad (\forall a \in D), \tag{3.2'}$$

$$\text{FAL } x,\ F_{q,1}(x) < F_{q,a}(x) \quad (\forall a \in F_q^*\backslash\{1\}), \tag{3.3'}$$

$$\text{FAL } x,\ F_{q,1}(x) > F_{q,a}(x) \quad (\forall a \in F_q^*\backslash\{1\}). \tag{3.4'}$$

Among the results of this section we show the existence of bounded barriers for (3.3') and (3.4') when $q \geqslant 7$, $q \notin \{8, 10, 12, 24\}$, and show that no finite barriers exist for (3.3') when $q \in \{8, 12, 24\}$. We also show that no bounded barriers exist for (3.3') and (3.4') when $q \in \{5, 10\}$.

For fixed q define the quantities (analogs of (1.6), (1.7))

$$\begin{aligned}
N(\rho,\chi) &= \text{ the multiplicity of the zero } \rho \text{ of } L(s,\chi), \\
G(\rho) = G(\rho; a, b) &= \sum_{\chi \in C_q} N(\rho,\chi)(\chi(a) - \chi(b)), \\
\sigma(a,b) &= \sup\{\Re\rho : G(\rho) \neq 0\}, \\
Z(\chi; a, b) &= \{\rho : L(\rho,\chi) = 0, G(\rho) \neq 0, \Re\rho = \sigma(a,b)\}, \\
Z(a,b) &= \bigcup_{\chi \in C_q(a,b)} Z(\chi; a, b).
\end{aligned}$$

The condition that $Z(a,b)$ is nonempty means that the supremum of the real parts of the zeros ρ of $L(s,\chi)$ with $\chi\in C_q(a,b)$ and $G(\rho)\neq 0$ is attained. In this case the sums over zeros in Corollary 1.3 are almost periodic functions in the Stepanov sense. In the case $b=1$, the condition $G(\rho)\neq 0$ is equivalent to the statement that $L(\rho,\chi)=0$ for some χ with $\chi(a)\neq 1$ (in fact $\Re G(\rho)<0$ in this case).

Theorem 3.1. *Suppose $q\geqslant 3$, $D\subset F_q^*$ and $1\notin D$. Suppose $\mathcal{B}$ is a system such that for each $a\in D$ the set $z(a,1)$ is nonempty. Then $\mathcal{B}$ is a barrier for the statement $\mathcal{I}(\mathcal{F})$:*

$$\textit{For sufficiently large } x,\ \exists a\in D: F_{q,1}(x)\geqslant \frac{F_{q,a}(x)+F_{q,a^{-1}}(x)}{2}.$$

Corollary 3.2. *Suppose $q\geqslant 3$, $D\subset F_q^*$, $1\notin D$, and for each $a\in D$, $a^2\equiv 1 \pmod q$. If $\mathcal{B}$ is a system such that $z(a,1)$ is nonempty for $a\in D$, then (3.1') holds. Consequently, there are no finite barriers for (3.3') when $q\in\{8,12,24\}$.*

Corollary 3.3. *Suppose $q\geqslant 3$, $D\subset F_q^*$ and $1\notin D$. If $Z(a,1)$ is non-empty for each $a\in D$, then*

$$\textit{FAL } x,\ \pi_{q,1}(x)<\frac{\pi_{q,a}(x)+\pi_{q,a^{-1}}(x)}{2}\qquad (\forall a\in D).$$

If in addition for each $a\in D$, $a^2\equiv 1 \pmod q$, then (3.1) holds. In particular, if $q\in\{8,12,24\}$ and $Z(a,1)$ is nonempty for $a\in F_q^*\setminus\{1\}$, then (3.3) holds.

Proof of Theorem 3.1. We have $C_q(a,1)=C_q(a^{-1},1)$ and $z(a,1)=z(a^{-1},1)$ for $a\in D$. For each $\chi\in C_q(a,1)$, $(\overline{\chi}(a)+\overline{\chi}(a^{-1}))/2-1=\Re\chi(a)-1$ is a negative real number. Let $\beta_a=\beta(a,1)$ for each $a\in D$ and put $\beta=R^-(\mathcal{B})$. Clearly $\beta\leqslant\min_{a\in D}\beta_a$. Let $\mathcal{F}$ be β-similar to $\mathcal{P}_q$. By (1.8) and (1.9), for each $a\in D$ we have as $u\to\infty$

$$\frac{u\phi(q)}{e^{\beta_a u}}\left(F_{q,1}(e^u)-\frac{F_{q,a}(e^u)+F_{q,a^{-1}}(e^u)}{2}\right)=-2\sum_{\chi\in C_q(a,1)}(1-\Re\chi(a))R_a(u;\chi)+o(1), \tag{3.5}$$

where

$$R_a(u;\chi)=\frac{n(\beta_a,\chi)}{2\beta_a}+\sum_{\substack{\gamma\in z(\chi;a,1)\\ \gamma>0}} n(\beta_a+i\gamma,\chi)\frac{\sin(\gamma u+\tan^{-1}(\beta_a/\gamma))}{\sqrt{\gamma^2+\beta_a^2}}. \tag{3.6}$$

Since each $z(a,1)$ is nonempty, it follows that for each $a\in D$ one of the functions $R_a(u;\chi)$ is not identically zero. Each function $R_a(u;\chi)$ is almost periodic in the sense of Bohr. To prove the theorem it suffices to show that there is a u for which each $R_a(u;\chi)>0$ (among those functions which are not identically zero). Clearly $u=0$ is such a number. □

Proof of Corollary 3.3. Let $\sigma_a = \sigma(a,1)$ for $a \in D$, $A_1 = \{a : \sigma_a > 1/2\}$, $A_2 = \{a : \sigma_a = 1/2\}$, $\beta = \min_{a \in A_1} \sigma_a$. For each $\chi \in C_q$, let $B(\chi)$ be the sequence of all zeros of $L(s,\chi)$ with real part $\geqslant \beta$, so $z_{\mathcal{B}}$ holds. If A_2 is empty, the Corollary follows from Lemma 1.4. Otherwise, by Lemma 1.1 we have for each $a \in A_2$,

$$\frac{u\phi(q)}{e^{u/2}}\left(\pi_{q,1}(e^u) - \frac{\pi_{q,a}(e^u)+\pi_{q,a^{-1}}(e^u)}{2}\right) = -2\sum_{\chi \in C_q(a,1)} (1-\Re\chi(a))R_a(u;\chi) + (N_q(a)-N_q(1)) + o(1) \quad (u \to \infty).$$

where

$$R_a(u;\chi) = N(1/2,\chi) + \sum_{\substack{L(1/2+it,\chi)=0\\ t>0}} N(1/2+it,\chi)\frac{\sin(tu+\tan^{-1}(1/2t))}{\sqrt{t^2+1/4}}.$$

We always have $N_q(a) \leqslant N_q(1)$. As in the proof of Lemma 3.1, each $R_a(u;\chi)$ is positive in a neighborhood of $u=0$ when $a \in A_1$. When $a \in A_2$, $R_a(u;\chi)$ is continuous on $(0,\log 2)$ and $R_a(u;\chi) \to +\infty$ as $u \to 0^+$ ([K1]; [K2], Lemma 2). Therefore if u is positive and sufficiently small, it is a continuity point for all $R_a(u;\chi)$ and each $R_a(u;\chi) > 0$. □

The next results address inequalities (3.1'), (3.2') when D is a cyclic subgroup of F_q^* or order 3.

Theorem 3.4. *Suppose $q \geqslant 3$ and $G = \{1,a,a^2\} \subset F_q^*$ is a cyclic group of order 3. Suppose $\mathcal{B}$ is a system such that the set $z(a,1)$ is nonempty and consists of numbers with imaginary part $\geqslant 2+\sqrt{3}$. Then $\mathcal{B}$ is a barrier for the statements:*

For sufficiently large x, $F_{q,1}(x) \geqslant \min(F_{q,a}(x), F_{q,a^2}(x)$,

For sufficiently large x, $F_{q,1}(x) \leqslant \max(F_{q,a}(x), F_{q,a^2}(x)$,

Corollary 3.5. *Suppose $q \geqslant 3$ and $G = \{1,a,a^2\} \subset F_q^*$ is a cyclic group of order 3. If $Z(a,1)$ is non-empty and, in the case $\sigma(a,1) > 1/2$, $Z(a,1)$ consists of numbers with imaginary part $\geqslant 2+\sqrt{3}$, then*

$$FAL\ x,\ \pi_{q,1}(x) < \min(\pi_{q,a}(x), \pi_{q,a^2}(x)),$$

$$FAL\ x,\ \pi_{q,1}(x) > \max(\pi_{q,a}(x), \pi_{q,a^2}(x)).$$

Corollary 3.5 can be deduced from Theorem 3.4 in the same way as we have proved Corollary 3.3.

Proof of Theorem 3.4. Let $\beta = \beta(a,1)$ and put $\beta_0 = R^-(\mathcal{B})$. Clearly $\beta_0 \leqslant \beta$. Let $\mathcal{F}$ be β_0-similar to $\mathcal{P}_q$. For $j = 1,2$ let $K_j = \{\chi \in C_q : \chi(a) = e(j/3)\}$. By (1.8) and (1.9), we have

$$\frac{\phi(q)u}{2e^{\beta u}}\left(F_{q,a^j}(e^u) - F_{q,1}(e^u)\right) = f(u) + (-1)^j g(u) + o(1), \quad (u \to \infty)$$

where

$$f(v)=\frac{3}{2}\sum_{\gamma\in z(a,1)}\frac{n(\gamma)}{\sqrt{\gamma^2+\beta^2}}\sin(\gamma v+\tan^{-1}\tfrac{\beta}{\gamma})=f_1(v)+f_2(v),$$

$$f_1(v)=\frac{3}{2}\sum_{\gamma\in z(a,1)}\frac{n(\gamma)}{\gamma^2+\beta^2}\gamma\sin(\gamma v),\quad f_2(v)=\frac{3}{2}\sum_{\gamma\in z(a,1)}\frac{n(\gamma)}{\gamma^2+\beta^2}\beta\cos(\gamma v),$$

$$n(\gamma)=\sum_{\chi\in K_1\cup K_2}n(\beta+i\gamma,\chi),$$

$$g(v)=\frac{\sqrt{3}}{2}\sum_{\gamma\in z(a,1)}\frac{m(\gamma)}{\sqrt{\gamma^2+\beta^2}}\cos(\gamma v+\tan^{-1}\tfrac{\beta}{\gamma})=g_1(v)-g_2(v),$$

$$g_1(v)=\frac{\sqrt{3}}{2}\sum_{\gamma\in z(a,1)}\frac{m(\gamma)}{\gamma^2+\beta^2}\gamma\cos\gamma v,\quad g_2(v)=\frac{\sqrt{3}}{2}\sum_{\gamma\in z(a,1)}\frac{m(\gamma)}{\gamma^2+\beta^2}\beta\sin\gamma v,$$

$$m(\gamma)=\sum_{j=1}^{2}(-1)^j\sum_{\chi\in K_j}n(\beta+i\gamma,\chi).$$

Since $\sum n(\gamma)/\sqrt{\gamma^2+\beta^2}$ converges and $|m(\gamma)|\leqslant n(\gamma)$, the series in the definitions of the functions f_1, f_2, g_1, and g_2 are uniformly convergent, and thus these functions are Bohr almost periodic. We need only find a single v for which $f(v)-g(v)$ and $f(v)+g(v)$ are both positive, and a single v for which $f(v)-g(v)$ and $f(v)+g(v)$ are both negative. Using the approximation of f_1, f_2, g_1, and g_2 by trigonometric polynomials and Lemma 2.1,

(3.7)

$$\begin{aligned}
&\|\max(|f_1|,|f_2|+|g_1|+|g_2|)-|f_2|-|g_1|-|g_2|\|_2\\
&\qquad\geqslant\|\max(|f_1|,|f_2|+|g_1|+|g_2|)\|_2-\|f_2\|_2-\|g_1\|_2-\|g_2\|_2\\
&\qquad\geqslant\|f_1\|_2-\|f_2\|_2-\|g_1\|_2-\|g_2\|_2\\
&\qquad=\sqrt{\frac{9}{8}\sum_{\gamma\in z(a,1)}\frac{\gamma^2n^2(\gamma)}{(\gamma^2+\beta^2)^2}}-\sqrt{\frac{9}{8}\sum_{\gamma\in z(a,1)}\frac{\beta^2n^2(\gamma)}{(\gamma^2+\beta^2)^2}}-\sqrt{\frac{3}{8}\sum_{\gamma\in z(a,1)}\frac{\gamma^2m^2(\gamma)}{(\gamma^2+\beta^2)^2}}\\
&\qquad\quad-\sqrt{\frac{9}{8}\sum_{\gamma\in z(a,1)}\frac{\beta^2m^2(\gamma)}{(\gamma^2+\beta^2)^2}}\geqslant\left(\sqrt{\frac{9}{8}}-\sqrt{\frac{3}{8}}\right)S_1-\left(\sqrt{\frac{9}{8}}+\sqrt{\frac{3}{8}}\right)S_2,
\end{aligned}$$

∎

where

$$S_1=\sqrt{\sum_{\gamma\in z(a,1)}\frac{\gamma^2n^2(\gamma)}{(\gamma^2+\beta^2)^2}},\quad S_2=\sqrt{\sum_{\gamma\in z(a,1)}\frac{\beta^2n^2(\gamma)}{(\gamma^2+\beta^2)^2}}.$$

Further,

$$S_2/S_1\leqslant\max_{\gamma\in z(a,1)}\beta/\gamma<\max_{\gamma\in z(a,1)}1/\gamma\leqslant 1/(2+\sqrt{3}).$$

Substituting the last inequality into (3.7), we obtain

$$\begin{aligned}&\| \max(|f_1|, |f_2| + |g_1| + |g_2|) - |f_2| - |g_1| - |g_2|\|_2 \\ &> \left(\sqrt{\frac{9}{8}} - \sqrt{\frac{3}{8}} - \frac{1}{2+\sqrt{3}}\left(\sqrt{\frac{9}{8}} + \sqrt{\frac{3}{8}}\right)\right) S_1 = 0.\end{aligned}$$

Therefore, there exists v_1 such that

$$\max(|f_1(v_1)|, |f_2(v_1)| + |g_1(v_1)| + |g_2(v_1)|) - |f_2(v_1)| - |g_1(v_1)| - |g_2(v_1)| > 0,$$

which is equivalent to
$|f_1(v_1)| > |f_2(v_1)|+|g_1(v_1)|+|g_2(v_1)|$. Observe that $f_1(-v_1) = -f_1(v_1)$, $|f_2(-v_1)| = |f_2(v_1)|$, $|g_1(-v_1)| = |g_1(v_1)|$,$|g_2(-v_1)| = |g_2(v_1)|$. Thus, one of the numbers $v \in \{v_1, -v_1\}$ satisfies the inequality

$$f_1(v) > |f_2(v)| + |g_1(v)| + |g_2(v)|, \tag{3.8}$$

and the other satisfies the inequality

$$-f_1(v) > |f_2(v)| + |g_1(v)| + |g_2(v)|. \tag{3.9}$$

The inequalities (3.8) and (3.9) imply $f(v) > |g(v)|$ and $f(v) < -|g(v)|$, respectively, as required. This completes the proof of the theorem. □

Remarks. R. Rumely [R] has computed the small zeros of L-functions modulo q (with imaginary part $\leqslant 2600$) $3 \leqslant q \leqslant 72$ and several larger q, and all such zeros lie on the critical line. Thus for such q the hypothesis in Corollary 3.5 about the imaginary parts of the zeros in $Z_q(a,b)$ is satisfied.

Two following statements complement Theorem 3.1 and Corollaries 3.2 and 3.3 for the problem of winning.

Theorem 3.6. *For any n there is an effectively computable number τ such that if $q \geqslant 3$, $D \subset F_q^*$, $1 \notin D$, $\mathcal{B}$ is a system such that for each $a \in D$ the set $z(a,1)$ is nonempty, $\bigcup_{a\in D} z(a,1)$ consists of numbers with imaginary part $\geqslant \tau$ and contains at most n elements, then $\mathcal{B}$ is a barrier for the statement*

$$\text{For sufficiently large } x,\ \exists a \in D : F_{q,1}(x) \leqslant \frac{F_{q,a}(x) + F_{q,a^{-1}}(x)}{2}.$$

Corollary 3.7. *Suppose $q \geqslant 3$, $D \subset F_q^*$, $1 \notin D$, and for each $a \in D$, $a^2 \equiv 1 \pmod q$. Then there are no bounded barriers for (3.2'). Consequently, there are no bounded barriers for (3.4') when $q \in \{8, 12, 24\}$.*

Corollary 3.8. *For any n there is an effectively computable number τ such that if $q \geqslant 3$, $D \subset F_q^*$, $1 \notin D$, for each $a \in D$ we have $a = a^{-1}$, for each $a \in D$ the set $Z(a,1)$ is nonempty, $\bigcup_{a \in D} Z(a,1)$ consists of numbers with imaginary part $\geqslant \tau$ and contains at most n elements, then (3.2) holds.*

Proof of Theorem 3.6. We have $C_q(a,1) = C_q(a^{-1},1)$ and $z(a,1) = z(a^{-1},1)$ for $a \in D$. For each $\chi \in C_q(a,1)$, $(\overline{\chi}(a) + \overline{\chi}(a^{-1}))/2 - 1 = \Re\chi(a) - 1$ is a negative real number. Let $\beta_a = \beta(a,1)$ for each $a \in D$ and put $\beta = R^-(\mathcal{B})$. Clearly $\beta \leqslant \min_{a \in D} \beta_a$. Let $\mathcal{F}$ be β-similar to $\mathcal{P}_q$. Take $\tau = 1/\varepsilon_2$, where $\varepsilon_2 = \varepsilon_2(n)$ was defined in Lemma 2.6. By (1.2) and Lemma 2.6, there exists a real number u such that $\sin(\gamma u + \tan^{-1}(\beta_a/\gamma)) < -\varepsilon_2$ for each $a \in D$ and $\gamma \in z(a,1)$. By periodicity of sines we can find an arbitrary large u satisfying these inequalities, and from (3.5) and (3.6) (notice that under our suppositions $n(\beta_a, \chi) = 0$) we deduce the assertion of the theorem. □

Theorem 3.9. *For any n there is an effectively computable number τ such that if $q \geqslant 5$, $G \subset F_q^*$ is a cyclic group of order 4, for each $a \in G \setminus \{1\}$ the set $z(a,1)$ is nonempty, $\bigcup_{a \in G\setminus\{1\}} z(a,1)$ consists of numbers with imaginary part $\geqslant \tau$ and contains at most n elements, then $\mathcal{B}$ is a barrier for the statements*

$$\text{For sufficiently large } x,\ \exists a \in G \setminus \{1\} : F_{q,1}(x) \geqslant F_{q,a}(x),$$

$$\text{For sufficiently large } x,\ \exists a \in G \setminus \{1\} : F_{q,1}(x) \leqslant F_{q,a}(x).$$

Consequently, there are no bounded barriers for (3.3') and (3.4') when $q \in \{5, 10\}$.

Corollary 3.10. *For any n there is an effectively computable number τ such that if $q \geqslant 5$, $G \subset F_q^*$ is a cyclic group of order 4, for each $a \in G \setminus \{1\}$ the set $Z(a,1)$ is nonempty, $\bigcup_{a \in G\setminus\{1\}} Z(a,1)$ consists of numbers with imaginary part $\geqslant \tau$ and contains at most n elements, then (3.1) and (3.2) hold for $D = G \setminus \{1\}$.*

Proof of Theorem 3.9. Let $G = \{1, a_1, a_2, a_3\}$, $a_j = a_1^j$ for $j = 2, 3$, $\beta_1 = \beta(a_1, 1)$, $\beta_2 = \beta(a_2, 1)$ and $\beta_0 = R^-(\mathcal{B})$. Clearly $\beta_0 \leqslant \beta_2 \leqslant \beta_1$. Let $\mathcal{F}$ be β_0-similar to $\mathcal{P}_q$. For $j = 1, 2, 3$ let $K_j = \{\chi \in C_q : \chi(a_1) = e(j/4)\}$. By (1.8) and (1.9), we have, as $u \to \infty$,

$$\frac{\phi(q)u}{2e^{\beta_1 u}} \left(F_{q,a_1}(e^u) - F_{q,1}(e^u)\right) = f(u) + g(u) + o(1), \tag{3.10}$$

$$\frac{\phi(q)u}{2e^{\beta_2 u}} \left(F_{q,a_2}(e^u) - F_{q,1}(e^u)\right) = h(u) + o(1), \tag{3.11}$$

$$\frac{\phi(q)u}{2e^{\beta_1 u}} \left(F_{q,a_3}(e^u) - F_{q,1}(e^u)\right) = f(u) - g(u) + o(1), \tag{3.12}$$

where

$$\begin{aligned}
f(v) &= \sum_{\gamma \in z(a_1,1)} \frac{k_1(\gamma)+2l(\gamma)}{\sqrt{\gamma^2+\beta_1^2}} \sin(\gamma v + \tan^{-1}\tfrac{\beta_1}{\gamma}),\\
g(v) &= \sum_{\gamma \in z(a_1,1)} \frac{m(\gamma)}{\sqrt{\gamma^2+\beta_1^2}} \cos(\gamma v + \tan^{-1}\tfrac{\beta_1}{\gamma}),\\
h(v) &= \sum_{\gamma \in z(a_2,1)} \frac{2k_2(\gamma)}{\sqrt{\gamma^2+\beta_2^2}} \sin(\gamma v + \tan^{-1}\tfrac{\beta_2}{\gamma}),\\
k_j(\gamma) &= \sum_{\chi \in K_1 \cup K_3} n(\beta_j + i\gamma, \chi) \quad (j=1,2),\\
l(\gamma) &= \sum_{\chi \in K_2} n(\beta_1 + i\gamma, \chi),\\
m(\gamma) &= \sum_{\chi \in K_1} n(\beta_1 + i\gamma, \chi) - \sum_{\chi \in K_3} n(\beta_1 + i\gamma, \chi).
\end{aligned}$$

Define $\varepsilon_2 = \varepsilon_2(n)$ from Lemma 2.6. We consider two cases.

Case I: $\beta_2 < \beta_1$. From (1.6) and (1.7), it follows that $k_1(\gamma) = m(\gamma) = 0$ for all γ. By Lemma 2.6 and the almost periodicity of $f(u)$ and $h(u)$, there are arbitrarily large u so that

$$f(u) < -2\varepsilon_2 \sum_{\gamma \in z(a_1,1)} \frac{l(\gamma)}{\sqrt{\gamma^2+\beta_1^2}}, \quad h(u) < -2\varepsilon_2 \sum_{\gamma \in z(a_2,1)} \frac{k_2(\gamma)}{\sqrt{\gamma^2+\beta_2^2}},$$

whence (3.2') holds with $D = G\backslash\{1\}$. Similarly, applying Lemma 2.6 to the functions $\sin(v - \tan^{-1}\beta_j/\gamma)$, there are arbitrarily large u so that

$$f(u) > 2\varepsilon_2 \sum_{\gamma \in z(a_1,1)} \frac{l(\gamma)}{\sqrt{\gamma^2+\beta_1^2}}, \quad h(u) > 2\varepsilon_2 \sum_{\gamma \in z(a_2,1)} \frac{k_2(\gamma)}{\sqrt{\gamma^2+\beta_2^2}},$$

whence (3.1') holds.

Case II: $\beta_2 = \beta_1$. Write $\beta = \beta_1 = \beta_2$. Here we have $z(a_2,1) \subseteq z(a_1,1) = z(a_3,1)$ and $k_1(\gamma) = k_2(\gamma)$. We again separate into two cases.

Case IIa: We have

$$\sum_{\gamma \in z(a_1,1)} \frac{|m(\gamma)|}{\sqrt{\gamma^2+\beta^2}} \leqslant \frac{\varepsilon_2}{2} \sum_{\gamma \in z(a_1,1)} \frac{k_1(\gamma)+2l(\gamma)}{\sqrt{\gamma^2+\beta^2}}. \tag{3.13}$$

By Lemma 2.6, there are arbitrarily large u so that

$$f(u) < -\varepsilon_2 \sum_{\gamma \in z(a_1,1)} \frac{k_1(\gamma)+2l(\gamma)}{\sqrt{\gamma^2+\beta_1^2}}, \quad h(u) < -2\varepsilon_2 \sum_{\gamma \in z(a_2,1)} \frac{k_2(\gamma)}{\sqrt{\gamma^2+\beta_2^2}}.$$

Since $|m(\gamma)| \leqslant k_1(\gamma)$, (3.13) implies that for such u, $|g(u)| < \frac{1}{2}|f(u)|$. Thus, by (3.10)–(3.12), (3.2') holds. Similarly, applying Lemma 2.6 to the functions $\sin(v - \tan^{-1}\beta/\gamma)$, we see that (3.1') holds.

Case IIb: (3.13) does not hold. By (3.10)–(3.12) and the almost periodicity of f, g, h, the theorem will follow if we show that there are real u and v such that

$$f(u) > \max(|g(u)|, f(u) - h(u)/2), \tag{3.14}$$

$$f(v) < \min(-|g(v)|, f(v) - h(v)/2). \tag{3.15}$$

We approximate $f, g, f - h/2$ by the polynomials

$$Q(u) = \sum_{\gamma \in z(a_1,1)} \frac{k_1(\gamma) + 2l(\gamma)}{\sqrt{\gamma^2 + \beta^2}} \sin(\gamma u),$$

$$P(u) = \sum_{\gamma \in z(a_1,1)} \frac{m(\gamma)}{\sqrt{\gamma^2 + \beta^2}} \cos(\gamma u),$$

$$R(u) = \sum_{\gamma \in z(a_1,1)} \frac{2l(\gamma)}{\sqrt{\gamma^2 + \beta^2}} \sin(\gamma u).$$

Note that $|m(\gamma)| \leqslant k_1(\gamma)$. Since (3.13) fails, we can use Lemma 2.7 with $\gamma = \varepsilon_2/2$. Thus, there exists a real number u_0 such that

$$Q(u_0) > \max(|P(u_0)|, R(u_0)) + \varepsilon S, \tag{3.16}$$

where $\varepsilon = \varepsilon_3 \gamma^2$, $S = \sum_\gamma \frac{k_1(\gamma)+2l(\gamma)}{\sqrt{\gamma^2+\beta^2}}$. The inequality (3.16) clearly implies

$$Q(-u_0) < \min(-|P(-u_0)|, R(-u_0)) - \varepsilon S. \tag{3.17}$$

Taking into account (1.2), we get

$$|f(u) - Q(u)| \leqslant S/\tau, \quad |g(u) - P(u)| \leqslant S/\tau, \quad |f(u) - h(u)/2 - R(u)| \leqslant S/\tau.$$

Therefore, we deduce (3.14) from (3.16) for $u = u_0$ and (3.15) from (3.17) for $v = -u_0$ provided that $2S/\tau < \varepsilon S$, or $\tau > 2/\varepsilon$. This completes the proof. $\square$

Theorem 3.11. *Let $q \geqslant 7$, $q \notin \{8, 10, 12, 24\}$. There is a set $D \in F_q^* \backslash \{1\}$ with $|D| = 3$ so that for any $\tau > 0$ there is a system $\mathcal{B}$ with $|\mathcal{B}| \leqslant 34$ which is a barrier for both inequalities (3.1') and (3.2'), and each sequence $B(\chi)$ consists of numbers with imaginary part $> \tau$;*

Proof. The argument depends on the group structure of F_q^*. Denote by Z_k the cyclic group of order k. Every F_q^*, $q \geqslant 7$, $q \notin \{8, 10, 12, 24\}$, either contains a cyclic group of even order $n \geqslant 6$ or contains a subgroup isomorphic to $Z_4 \times Z_2$. Our constructions depend on properties of the functions

$$Q(v) = 2\sin v + \frac{1}{2}\sin(6v),$$
$$P(v) = 2\cos v - \frac{1}{2}\cos(6v),$$
$$R(v) = \sum_{k=2}^{7} \frac{p_k}{k}\sin(kv), \quad p_2 = 1, p_3 = 2, p_4 = 3, p_5 = 4, p_6 = 3, p_7 = 2.$$

The critical properties are

$$|P(v)| > \sqrt{3}Q(v) \quad (0 \leqslant v \leqslant 0.759, 2.7 \leqslant v \leqslant 2\pi), \qquad (3.18)$$
$$R(v) < 0 \quad (0.758 \leqslant v < \pi).$$

We first consider the case when F_q^* has an element of even order n with $n \geqslant 6$. Without loss of generality, if n is a power of 2, assume $n = 8$. Let χ be a character of order n, and let a be an element of F_q^* of order n such that $\chi(a) = e(-1/n)$. Fix $\beta > \frac{1}{2}$ and large $\gamma > 0$, let $n(\beta + ik\gamma, \chi^j) = m_{j,k}$ $(1 \leqslant j \leqslant n-1,\ 1 \leqslant k \leqslant K)$. Suppose $n(\rho, \chi) = 0$ for all other pairs (ρ, χ). Let $\mathcal{F}$ be β-similar to $\mathcal{P}_q$. By (1.8) and (1.9), as $u \to \infty$,

$$\frac{\phi(q)u\gamma}{2e^{\beta u}}\left(F_{q,a^r}(e^u) - F_{q,1}(e^u)\right) = G_0(u\gamma) - G_r(u\gamma) + O\left(\frac{1}{\gamma} + \frac{1}{u}\right) + o(1),$$

where

$$G_r(v) = \sum_{j,k} \frac{m_{j,k}}{k}\sin\left(kv + \frac{2\pi jr}{n}\right)$$
$$= \sum_{j,k} \frac{m_{j,k}}{k}\left[\cos\left(\frac{2\pi jr}{n}\right)\sin(kv) + \sin\left(\frac{2\pi jr}{n}\right)\cos(kv)\right].$$

We take $D = \{a^s, a^{n-s}, a^{n/2}\}$ for some $s \neq n/2$. The theorem will follow if we show that for every $v \in [0, 2\pi)$, there is a $r \in \{s, n-s, n/2\}$ so that $G_r(v) > G_0(v)$, since $G_r(v) > G_0(v)$ implies $G_{n-r}(-v) > G_0(-v)$.

First, if $n = 2^d h$, where h is odd and $h \geqslant 3$, we take $m_{2,6} = 3$, $m_{2h-2,1} = 2$, $m_{h,k} = p_k$ for $2 \leqslant k \leqslant 7$ and $m_{j,k} = 0$ for other j, k, so $|\mathcal{B}| = 20$. We obtain

$$G_0(v) - G_{2^d}(v) = (1 - \cos(4\pi/h))Q(v) + \sin(4\pi/h)P(v),$$
$$G_0(v) - G_{n-2^d}(v) = (1 - \cos(4\pi/h))Q(v) - \sin(4\pi/h)P(v),$$
$$G_0(v) - G_{n/2}(v) = 2R(v).$$

The theorem follows in this case from (3.22) and the fact that

$$\left|\frac{1 - \cos(4\pi/h)}{\sin(4\pi/h)}\right| \leqslant \sqrt{3}.$$

Next, suppose $n = 8$ and take $m_{2,1} = 4$, $m_{3,k} = m_{5,k} = p_k$ for $2 \leqslant k \leqslant 7$ and $m_{j,k} = 0$ for other j, k, so $|\mathcal{B}| = 34$. Then

$$\begin{aligned} G_0(v) - G_3(v) &= 4(\sin v - \cos v) + (2 - \sqrt{2})R(v), \\ G_0(v) - G_5(v) &= 4(\sin v + \cos v) + (2 - \sqrt{2})R(v), \\ G_0(v) - G_4(v) &= 4R(v). \end{aligned}$$

When $0 \leqslant v \leqslant 0.758$ or $\pi \leqslant v \leqslant 2\pi$, one of the first two functions is negative.

The last case is when F_q^* has a subgroup G isomorphic to $Z_4 \times Z_2$. Let $\{a, b\}$ generate G, a having order 4 and b having order 2. Let χ_1 have order 4, χ_2 have order 2 so that

$$\chi_1(a) = -i, \chi_1(b) = 1, \quad \chi_2(a) = 1, \chi_2(b) = -1.$$

Fix $\beta > \frac{1}{2}$ and large $\gamma > 0$, and let, for some L, $n(\beta + il\gamma, \chi_1^j \chi_2^k) = m_{j,k,l}$ for $0 \leqslant j \leqslant 3, 0 \leqslant k \leqslant 1, (j,k) \neq (0,0)$, $1 \leqslant l \leqslant L$. Suppose $n(\rho, \chi) = 0$ for all other pairs (ρ, χ). Let $\mathcal{F}$ be β-similar to $\mathcal{P}_q$. By (1.8) and (1.9), as $u \to \infty$,

$$\frac{\phi(q)u\gamma}{2e^{\beta u}} \left(F_{q,a^r b^s}(e^u) - F_{q,1}(e^u)\right) = G_{0,0}(u\gamma) - G_{r,s}(u\gamma) + O\left(\frac{1}{\gamma} + \frac{1}{u}\right) + o(1),$$

where

$$G_{r,s}(v) = \sum_{j,k,l} \frac{m_{j,k,l}}{l} \sin\left(lv + \frac{\pi}{2} rj + \pi sk\right).$$

Note that $G_{0,0}(v) < G_{r,s}(v)$ implies $G_{0,0}(-v) > G_{4-r,2-s}(-v)$. We take $D = \{a, a^3, b\}$. Thus, if for all $v \in [0, 2\pi)$, $G_{0,0}(v) < G_{r,s}(v)$ for some pair $(r,s) \in \{(1,0), (3,0), (0,1)\}$, then $\mathcal{B}$ is a barrier for both (3.1') and (3.2'). We take $m_{1,0,1} = 1$ ($L(s, \chi_1)$ has a simple zero at $s = \beta + it$), and $m_{0,1,l} = p_l$ for $2 \leqslant l \leqslant 7$, Take $m_{j,k,l} = 0$ for other (j,k,l), so $|\mathcal{B}| = 16$. Then

$$\begin{aligned} G_{0,0}(v) - G_{1,0}(v) &= \sin v - \cos v, \\ G_{0,0}(v) - G_{3,0}(v) &= \sin v + \cos v, \\ G_{0,0}(v) - G_{0,1}(v) &= 2R(v). \end{aligned}$$

When $0 \leqslant v < \pi/4$ or $3\pi/4 < v \leqslant 2\pi$, we have $|\cos v| > \sin v$, whence either $G_{0,0}(v) - G_{1,0}(v) < 0$ or $G_{0,0}(v) - G_{3,0}(v) < 0$. For the remaining v, $R(v) < 0$ by (3.18). □

Corollary 3.12. *Let $q = 5$ or $q \geqslant 7$. Each inequality (3.3'), (3.4') possesses a bounded barrier if and only if $q \notin \{5, 8, 10, 12, 24\}$.*

4. Extremal Barriers

By an *ordering* of the functions $\pi_{q,a_i}(x)$ $(1 \leqslant i \leqslant r)$ we mean a chain of inequalities

$$\pi_{q,a_{i(1)}}(x) \geqslant \pi_{q,a_{i(2)}}(x) \geqslant \ldots \geqslant \pi_{q,a_{i(r)}}(x),$$

where $\{i(1), \ldots, i(r)\}$ is a permutation of $\{1, \ldots, r\}$. Thus, we admit non-strict inequalities in orderings, and in the case of coincidence of some functions $\pi_{q,a_i}(x)$ several orderings occur for x. Let $S_q(D)$ be the number of orderings of the functions $\pi_{q,a}(x)$ $(a \in D)$ which occur for arbitrarily large x. Likewise, for a system $\mathcal{B}$ and set of functions $\mathcal{F}$, define $s(D) = s(D; \mathcal{F})$ to be the number of orderings of functions $F_{q,a}(x; \mathcal{B})$ $(a \in D)$ which occur for arbitrarily large x.

If $\pi_{q,a}(x) > \pi_{q,b}(x)$ and $\pi_{q,a}(y) < \pi_{q,b}(y)$, then $\pi_{q,a}(w) = \pi_{q,b}(w)$ at some point w between x and y. This property of these functions is crucial to results about $S_q(D)$. If a set of functions $\mathcal{F}$ has the property that for $f_i, f_j \in \mathcal{F}$, $f_i(x) < f_j(x)$ and $f_i(y) > f_j(y)$ implies $f_i(w) = f_j(w)$ for some w between x and y, we say that $\mathcal{F}$ is *good.*

Let $D \subseteq F_q^*$ and $\beta = R^-(\mathcal{B})$. We say that $\mathcal{B}$ is a *KT-system* (Knapowski-Turán system) for D, if for each set of functions $\mathcal{F}$ which is β-similar to $\mathcal{P}_q$ and every distinct $a, b \in D$,

$$\text{FAL } x,\ F_{q,a}(x) > F_{q,b}(x).$$

If $\mathcal{B}$ is a KT-system for D and $z_{\mathcal{B}}$ holds, then each difference $\pi_{q,a}(x) - \pi_{q,b}(x)$, $a, b \in D$, changes sign infinitely often. For several moduli q this is known unconditionally for all differences $\pi_{q,a}(x) - \pi_{q,b}(x)$, $a, b \in F_q^*$, $a \neq b$ (see [FK2]). A KT-system D has the property that for distinct $a, b \in D$ there is some ρ with $g(\rho; a, b) \neq 0$, for otherwise $D_{q,a,b}(x)$ is identically zero and one could take $F_{q,c}(x) = P_{q,c}(x; \mathcal{B})$ for each $c \in D$. In the opposite direction we have the following.

Proposition 4.1. *Let $D \subseteq F_q^*$. Every system $\mathcal{B}$ which lacks real elements and for which $z(a, b)$ is nonempty for $a, b \in D$ is a KT-system for D.*

Proof. Take distinct $a, b \in D$, let $\beta = R^-(\mathcal{B})$ and suppose $\mathcal{F}$ is β-similar to $\mathcal{P}_q$. By (1.5), (1.6), (1.8) and (1.9), as $u \to \infty$

$$\frac{u\phi(q)}{2e^{\beta u}} \left(F_{q,a}(e^u) - F_{q,b}(e^u)\right) = -h(u) + o(1), \tag{4.1}$$

where $\beta = \beta(a, b)$ and

$$h(u) = \sum_{\beta + i\gamma \in z(a,b)} \Re \left(\overline{g(\beta + i\gamma)} \frac{e^{i\gamma u}}{\beta + i\gamma} \right). \tag{4.2}$$

By (1.4), the partial sums of (4.2) uniformly converge to h. By Lemma 2.1,

$$\sup_u h(u) \geqslant \sup_\gamma \frac{|g(\beta + i\gamma)|}{4|\beta + i\gamma|},$$
$$\sup_u -h(u) \geqslant \sup_\gamma \frac{|g(\beta + i\gamma)|}{4|\beta + i\gamma|}.$$

Taking into account that h is almost periodic function in the Bohr sense, we get from (4.1)

$$\liminf_{x\to\infty} \frac{\log x}{x^\beta} \left(F_{q,a}(x) - F_{q,b}(x)\right) < 0,$$

$$\limsup_{x\to\infty} \frac{\log x}{x^\beta} \left(F_{q,a}(x) - F_{q,b}(x)\right) > 0,$$

and the proposition is proved. □

Theorem 4.2. *If $\mathcal{B}$ is a KT-system for $D = \{a_1, a_2, \dots, a_r\}$, then for every good $\mathcal{F}$ which is β-similar to $\mathcal{P}_q$ ($\beta = R^-(\mathcal{B})$), at least $r(r-1)/2+1$ orderings of the functions $F_{q,a_i}(x)$ occur for arbitrarily large x. Consequently, under the condition $z_{\mathcal{B}}$, $S_q(D) \geqslant r(r-1)/2+1$.*

Proof. Fix a good $\mathcal{F}$ which is β-similar to $\mathcal{P}_q$. Let us construct a graph G. For each permutation $P = (i(1), \dots, i(r))$ of the set $\{1, \dots, r\}$, let $N(P)$ be the set of real $x \geqslant 1$ with

$$F_{q,a_{i(1)}}(x) \geqslant F_{q,a_{i(2)}}(x) \geqslant \dots \geqslant F_{q,a_{i(r)}}(x).$$

For each unbounded set $N(P)$, associate a vertex $v(P)$ of G. Put an edge from $v(P_1)$ to $v(P_2)$ whenever (i) P_2 is obtained from P_1 by transposing two neighbor elements k, l, and (ii) $N(P_1) \cap N(P_2)$ is unbounded (note $x \in N(P_1) \cap N(P_2)$ implies $F_{q,k}(x) = F_{q,l}(x)$). Label such an edge by $\{k, l\}$.

Also, as $\mathcal{B}$ is KT-system and $\mathcal{F}$ is good, for any numbers i and j, $1 \leqslant i < j \leqslant r$, there is an edge labeled by (i,j). We claim that the graph G contains a subgraph G' such that each component of G' is a tree and the labelings of the edges in G' contain again all possible pairs (i,j). Indeed if G contains a cycle H take two vertices g_1 and g_2 from H. Then there are numbers i and j occurring in g_1 and g_2 in opposite orders. This means that in both arcs of the cycle H connecting g_1 and g_2 there is an edge labeled by (i,j). Delete one of them. We can repeat this procedure as long as the remaining graph contains at least one cycle. In the end we get a required subgraph G'. The number of edges of G' is at least the number of distinct labels, thus it is at least $r(r-1)/2$. Therefore, the number of vertices of G' is $\geqslant r(r-1)/2+1$. □

A system $\mathcal{B}$ is called an *extremal barrier* for D if it is a KT-system for D and a barrier for the statement

$$s(D) \geqslant \frac{r(r-1)}{2} + 2.$$

By Lemma 1.4, if $\mathcal{B}$ is an extremal barrier and $z_{\mathcal{B}}$ holds, at most $r(r-1)/2+1$ orderings of the functions $\pi_{q,a}(x)$ ($a \in D$) occur for large x. An interesting problem is to describe for each q the sets D possessing finite extremal barriers. We are very far from a complete solution to this problem; in particular, there is no q, $\varphi(q) > 2$, for which we know whether the whole system F_q^* has a finite extremal barrier. In this section we present some results on existence and nonexistence of extremal barriers. In particular we shall see that for large moduli q there is a finite extremal barrier for some set D with $|D| = r(q) \to \infty$ as $q \to \infty$.

Theorem 4.3. *For every cyclic group $G \subset F_q^*$ of order $r \geqslant 6$ and for every set $D \subset G$ such that $1 \notin D$ and $a^{-1} \notin D$ if $a^{-1} \neq a \in D$, there is a bounded extremal barrier for D.*

Remark. *The size of $\mathcal{B}$ in our construction depends only on r, and it can be effectively computed.*

To prove Theorem 4.3, we take a generator a_1 of the group G and a character χ_1 so that $\chi_1(a_1) = e(-1/r)$. For $j = 1, \dots, r-1$ denote $a_j = a_1^j$, $\chi_j = \chi_1^j$. Take $\beta_1 \in (1/2, 1)$, large γ and large positive integer K depending on r. The idea is to put $n(\beta_1 + ki\gamma, \chi_j) = N_{k,j}$ ($k = 1, \dots, K$, $j = 1, \dots, r-1$), where $N_{k,j}$ are appropriate nonnegative integers. For $k = 1, \dots, K$, $v = 0, \dots, r-1$ define the functions

$$G_{k,v}(u) = \sum_{j=1}^{r-1} \frac{N_{k,j}}{k} \sin(ku + 2\pi jv/r).$$

If $\mathcal{F}$ is β-similar to $\mathcal{P}_q$, we have for $1 \leqslant v, w < r$

$$F_{q,a_v}(x) - F_{q,a_w}(x) = \frac{2x^{\beta_1}}{\gamma \log x} \times$$

$$\left(\sum_{k=1}^{K} G_{k,v}(\gamma \log x) - G_{k,w}(\gamma \log x) + O\left(\frac{1}{\gamma}\right) \right) + o(1) \quad (x \to \infty). \tag{4.3}$$

To choose multiplicities $N_{k,j}$ we need the following lemma.

Lemma 4.4. *Let c_v, d_v ($v = 0, \dots, r-1$) be real numbers such that $c_v = c_{r-v}$ ($v = 1, \dots, r-1$), $d_0 = 0$, $d_v = -d_{r-v}$ ($v = 1, \dots, r-1$). Then there exist real numbers ν_j ($j = 0, \dots, r-1$) such that*

$$\sum_{j=0}^{r-1} \nu_j \sin(u + 2\pi jv/r) = c_v \sin u + d_v \cos u \quad (v = 0, \dots, r-1). \tag{4.4}$$

Proof. The system (4.4) is equivalent to the system of two systems of linear equations

$$\sum_{j=0}^{[r/2]} \mu_j \cos(2\pi jv/r) = c_v \quad (v = 0, \dots, [r/2]), \tag{4.5}$$

$$\sum_{j=1}^{[(r-1)/2]} \lambda_j \sin(2\pi jv/r) = d_v \quad (v = 1, \dots, [(r-1)/2]). \tag{4.6}$$

where $\mu_0 = \nu_0$, $\mu_j = \nu_j + \nu_{r-j}$, $\lambda_j = \nu_j - \nu_{r-j}$ ($1 \leqslant j \leqslant (r-1)/2$), $\nu_{r/2} = \mu_{r/2}$ if r is even. To prove solubility of the system (4.5) it suffices to check that the system has no nontrivial solutions for $c_v = 0$ ($v = 1, \dots, [r/2]$). Assume the contrary. Consider the trigonometric polynomial

$$T(u) = \sum_{j=0}^{[r/2]} \mu_j \cos(2\pi j u).$$

If not all μ_j are zero, the polynomial T has at most $2[r/2]$ zeros on $[0, 2\pi)$ counting with multiplicity. On the other hand, by (4.5) with our supposition $c_v = 0$, the points $2\pi v/r$ ($v = 0, \dots, r-1$) are zeros of T, and, moreover, 0 is a double zero. Hence, the total number of the zeros of T on $[0, 2\pi)$ counting with multiplicity is at least $r + 1 > 2[r/2]$. This contradiction shows that $T \equiv 0$. So, the system (4.5) has a unique solution for any c_v. In the same way we can prove the solubility of the system (4.6).

Now we have the existence of numbers μ_j and λ_j satisfying (4.5) and (4.6). To complete the proof of Lemma 4.4, it remains to set $\nu_{r/2} = \mu_{r/2}$ for even r, $\nu_0 = \mu_0$, $\nu_j = (\mu_j + \lambda_j)/2$ for $1 \leqslant j < r/2$, $\nu_j = (\mu_{r-j} - \lambda_{r-j})/2$ for $r/2 < j < r$. □

Here we shall apply Lemma 4.4 for the case $c_v = 0$ ($v = 0, \dots, r-1$). We have stated it for arbitrary c_v taking into account other applications.

Let $V = \{v : a_v \in D\}$. Let us take a system of continuous even 2π-periodic functions f_v, $v \in V$, and let us require the following properties to hold:
1) If $r/2 \in V$ then $f_{r/2} \equiv 0$;
2) $\int_0^\pi f_v(u)du = 0$ for all $v \in V$;
3) for every distinct $v \in V$ and $w \in V$, $v \neq w$, there is the unique point $u = u_{v,w} \in [0, \pi]$ at which $f_v(u) = f_w(u)$, and, moreover, for distinct (nonordered) pairs (v, w) the points $u_{v,w}$ are distinct.
Clearly, a system $\Omega = \{f_v\}$ exists; for example we can take several functions in a general position from the set of piecewise linear functions with zero average and one corner on $(0, \pi)$, with slope 0 to the right of 0 and slope 1 to the left of π.

Observe that the ordering of the functions $\{f_v(u)\}$, $u \in [0, \pi]$, changes only at points $u_{v,w}$. On the other hand, if points $u_1 \in [0, \pi]$ and $u_2 \in [0, \pi]$ are separated by some point $u_{v,w}$, then $(f_v(u_1) - f_w(u_1))(f_v(u_2) - f_w(u_2)) < 0$. Thus, the number of orderings of the functions $\{f_v(u)\}$, $u \in [0, \pi]$, is $|V|(|V| - 1)/2 + 1$. Since the functions f_v are even and 2π – *periodic*, this is the number of orderings on the whole real line.

Take U which is a multiple of 2π and arrange all the points $\pm u_{v,w} + 2\pi k \in (U, \infty)$ in increasing order $U < u_1 < u_2 < \dots$. Denote $u'_0 = U$, $u'_j = (u_j + u_{j+1})/2$ for $j \geqslant 1$. We say that a system $\tilde{\Omega}$ of continuous real functions $\tilde{f}_v(u)$, $u \in [U, \infty]$, is of Ω-type if for any $j \geqslant 0$ and for any $u \in [u'_j, u'_{j+1}]$ the ordering of the functions $\{\tilde{f}_v(u)\}$ coincides with the ordering $\{f_v(u'_j)\}$ or with the ordering $\{f_v(u'_{j+1})\}$ (recall that in the case of some equalities $\tilde{f}_v(u) = \tilde{f}_w(u)$ we assign to the point u several orderings). The system Ω is an example of a system of Ω-type. We will repeatedly use the following simple fact.

Proposition 4.5. *If a system $\tilde{\Omega}$ of 2π-periodic functions is of Ω-type, then every system sufficiently close to $\tilde{\Omega}$ in the uniform metric is of Ω-type. Moreover, every* *system whose pairwise differences are close to corresponding differences for $\tilde{\Omega}$ in the uniform metric is of Ω-type.*

Eventually, we shall show that the system $\{F_{q,a_v}(\gamma \log x)\}$, $v \in V$, is of Ω-type, which proves Theorem 4.3.

First, take any admissible system $\Omega = \{f_v : v \in V\}$. We approximate the functions f_v by even trigonometric polynomials T_v with zero average in the uniform norm. In the case $r/2 \in V$ we take $T_{r/2} \equiv 0$. Let

$$T_v(u) = \sum_{k=1}^{K} b_{k,v} \cos(ku).$$

By Proposition 4.5, for sufficiently large $K = K(r)$ we can make the approximation so good that the system $\{T_v\}$ is of Ω-type.

By the conditions on D, $r - v \notin V$ if $v \in V$ and $v \neq r/2$. Let $b_{k,r-v} = b_{k,v}$ for $v \in V$, and set $b_{k,v} = 0$ for $v \notin V$ and $r - v \notin V$. By Lemma 4.4, there exist real numbers $\nu_{k,j}$ $(k = 1, \dots, K,\ j = 0, \dots, r-1)$ such that

$$\sum_{j=0}^{r-1} \nu_{k,j} \sin(ku + 2\pi jv/r) = b_{k,v} \cos ku \quad (k = 1, \dots, K; v = 0, \dots, r-1).$$

Therefore,

$$T_v(u) = \sum_{k=1}^{K} \sum_{j=0}^{r-1} \nu_{k,j} \sin(ku + 2\pi jv/r) \quad (v \in V).$$

Take a positive integer N and define trigonometric polynomials

$$\tilde{T}_v(u) = \sum_{k=1}^{K} \sum_{j=0}^{r-1} \frac{\tilde{N}_{k,j}}{kN} \sin(ku + 2\pi jv/r) \quad (v \in V),$$

where

$$\tilde{N}_{k,j} = k[N\nu_{k,j}] \quad (k = 1, \dots, K; j = 1, \dots, r-1).$$

By Proposition 4.5, the system $\{\tilde{T}_v\}$ is of Ω-type provided that N is large enough.

Finally, take $\tilde{N} = \min_{k,j} \tilde{N}_{k,j}$, $N_{k,j} = \tilde{N}_{k,j} - \tilde{N} \geqslant 0$. Since $1 \notin D$, we have $0 \notin V$ and hence,

$$\sum_{j=0}^{r-1} \sin(ku + 2\pi jv/r) = 0 \quad (k = 1, \dots, K; v \in V)$$

and

$$N\tilde{T}_v(u) = \sum_{k=1}^{K} G_{k,v}(u) \quad (v \in V).$$

The equality (4.3) can be rewritten for $v, w \in V$ as

$$\frac{\gamma \log x}{2Nx^{\beta_1}}(F_{q,a_v}(x) - F_{q,a_w}(x)) = \tilde{T}_v(\gamma \log x) - \tilde{T}_w(\gamma \log x) + O\left(\frac{1}{\gamma}\right) + o(1) \quad (x \to \infty).$$

By Proposition 4.5, the system $\{\frac{\gamma \log x}{2Nx^{\beta_1}} F_{q,a_v}(x)\}$, $v \in V$, is of Ω-type on $[U, \infty)$ if U and γ are large enough. So is the system $\{F_{q,a_v}(x)\}$, $v \in V$, as required. □

It is not difficult to see that $\lambda(q) \to \infty$ as $q \to \infty$. A lower estimate

$$\lambda(q) > (\log q)^{c \log\log\log(q+20)}$$

with some $c > 0$ was established in [EPS]. Thus, we have the following.

Corollary 4.6. *For sufficiently large q there is a finite extremal barrier for some set D with $|D| = r(q) \geqslant \lambda(q)/2 \to \infty$ as $q \to \infty$.*

It is naturally to ask if there are bounded extremal barriers for $D = G$. We show that it is not so in the case $|G| = 3$. However, we cannot prove that for $|G| = 3$ there are no finite extremal barriers.

Theorem 4.7. *For any n there is an effectively computable number τ such that the following holds. Let $q \in \mathbf{N}$, $a \in F_q^*$, $a^3 = 1$, $G = \{1, a, a^2\}$. If $\mathcal{B}$ is a system such that $Z_q(a, 1)$ and $Z_q(a, a^2)$ are nonempty and $Z_q(a, 1) \cup Z_q(a, a^2)$ consists of numbers with imaginary part $\geqslant \tau$ and contains at most n elements, then $\mathcal{B}$ is a barrier for the statement*

$$\mathcal{F} \quad \textit{is good and} \quad s(\{1, a, a^2\}; \mathcal{F}) \leqslant 4.$$

Consequently, under the condition $z_{\mathcal{B}}$, $S_q(G) \geqslant 5$.

To prove Theorem 4.7, we first estimate the number of orderings if each of three players leads and trails for arbitrarily large x.

Lemma 4.8. *Let $D = \{a_1, a_2, a_3\} \subset F_q^*$ and $\mathcal{B}$ be such a system that for any function system $\mathcal{F}$ which is β-similar to $\mathcal{P}_q$ ($\beta = R^-(\mathcal{B})$), and for any $a' \in D$ there are arbitrary large x and y such that*

$$F_{q,a'}(x) > \max(F_{q,a''}(x) : \ a'' \in D \setminus \{a'\}),$$

$$F_{q,a'}(y) < \min(F_{q,a''}(y) : \ a'' \in D \setminus \{a'\}).$$

Then for any good function system $\mathcal{F}$ which is β-similar to $\mathcal{P}_q$, at least 5 orderings of the functions $\{F_{q,a'}(x) : a' \in D\}$ occur for arbitrary large x.

Proof. By Theorem 4.2, since $\mathcal{B}$ is a KT-system for D, at least 4 orderings occur for arbitrary large x. Assume only 4 orderings occur. Since $\mathcal{F}$ is good, there are arbitrary large x such that $F_{q,a_1}(x) \geqslant F_{q,a_2}(x) = F_{q,a_3}(x)$ or $F_{q,a_1}(x) \leqslant F_{q,a_2}(x) = F_{q,a_3}(x)$. Thus, in both cases for large x there are at least 3 orderings where a_1 leads or trails, and, therefore, at most one ordering where a_1 is in the second position.

The same holds for a_2 and a_3. Hence, the number of orderings where some player is in the second position, which is clearly the number of all orderings, is at most 3, but that is impossible. Lemma 4.8 is proved. □

Proof of Theorem 4.7. Let $\beta_0 = R^-(\mathcal{B})$ and a system $\mathcal{F}$ be good and β_0-similar to $\mathcal{P}_q$. Re-denote by τ' the number τ from Theorem 3.6. Take

$$\tau = \max(\tau', 1/\varepsilon_1(n)),$$

where $\varepsilon_1(n)$ is the number from Lemma 2.4, and suppose that the conclusion of the theorem does not hold. Then, by Lemma 4.8, one of the players $1, a, a^2$ does not lead FAL x or does not trail FAL x. We see from Theorem 3.4 that this is not player 1. Without loss of generality, assume that a^2 does not lead FAL x. Thus, the orderings

$$F_{q,a^2}(x) > F_{q,a}(x) \geqslant F_{q,1}(x), \quad F_{q,a^2}(x) > F_{q,1}(x) \geqslant F_{q,a}(x)$$

do not occur for large x.

We use notation and relationships form the proof of Proposition 4.1 with $b = a^2$. Note that for any $\chi \in C_q$ we have $\chi(a^2) = \overline{\chi(a)}$. Thus, $g(\rho) = g(\rho; a, a^2)$ is a purely imaginary number, and

$$\Re\left(\overline{g(\beta+i\gamma)}\frac{e^{i\gamma u}}{\beta+i\gamma}\right) = \frac{\overline{g(\beta+i\gamma)}}{i\sqrt{\gamma^2+\beta^2}}\cos(\gamma u + \tan^{-1}(\beta/\gamma)). \tag{4.7}$$

Now let us follow again the proof of Proposition 4.1 to approximate $P_{q,1}(e^u) - (P_{q,a}(e^u) + P_{q,a^2}(e^u))/2$. Let

$$g_1(\rho) = \sum_{\chi} n(\rho,\chi)(1 - (\chi(a) + \chi(a^2))/2),$$

$$\beta_1 = \max\{\Re\rho : \ g_1(\rho) \neq 0\},$$
$$\mathcal{R}_1 = \{\rho : \ \Re\rho = \beta_1, \ g_1(\rho) \neq 0\},$$
$$h_1(u) = \sum_{\beta_1+i\gamma\in\mathcal{R}_1} \Re\left(\overline{g_1(\beta_1+i\gamma)}\frac{e^{i\gamma u}}{\beta_1+i\gamma}\right).$$

The formula (4.1) written for $(1,a)$ and $(1,a^2)$ gives

$$\frac{u\phi(q)}{2e^{\beta_1 u}}\left(P_{q,1}(e^u) - (P_{q,a}(e^u) + P_{q,a^2}(e^u))/2\right) = -h_1(u) + o(1) \quad (u\to\infty). \tag{4.8}$$

Now, $g_1(\rho)$ is always a real number, and therefore

$$\Re\left(\overline{g_1(\beta_1+i\gamma)}\frac{e^{i\gamma u}}{\beta_1+i\gamma}\right) = \frac{g(\beta_1+i\gamma)}{\sqrt{\gamma^2+\beta_1^2}}\sin(\gamma u + \tan^{-1}(\beta_1/\gamma)). \tag{4.9}$$

Note, that in the definitions of h and h_1 the sum is taken over $\gamma \in Z_q(a, a^2)$ and, respectively, over $\gamma \in Z_q(a, 1)$. By the choice of τ and (1.2), any $\gamma \in Z_q(a, 1) \cup Z_q(a, a^2)$ satisfies the inequalities $\tan^{-1}(\beta/\gamma)) < \varepsilon_1$, $\tan^{-1}(\beta_1/\gamma)) < \varepsilon_1$. By the suppositions of the theorem, h and h_1 are nonzero polynomials with at most n distinct frequences γ in total. Therefore, we can apply Lemma 2.4 to h and h_1. Hence, there exist $\delta > 0$ and $u \in \mathbf{R}$ such that

$$h(u) > \delta, \quad h_1(u) > \delta. \tag{4.10}$$

As the functions h and h_1 are almost periodic in the Bohr sense, we can find an arbitrary large u satisfying (4.10). Then, by (4.1) and (4.8), and the β_0-similarity of the system $\mathcal{F}$ to $\mathcal{P}_q$, taking into account that $\beta_0 \leqslant \min(\beta, \beta_1)$, we get

$$F_{q,a^2}(x) > F_{q,a}(x), \quad (F_{q,a}(x) + F_{q,a^2}(x))/2 > F_{q,1}(x).$$

This contradicts our assumption that a^2 does not lead FAL x and completes the proof of Theorem 4.7. $\square$

5. The number of possible orderings

Theorem 5.1. *Fix q and an arbitrarily large τ. There is a system $\mathcal{B}$ satisfying*

(i) $|\mathcal{B}|$ bounded in terms of q;

(ii) $\rho \in \mathcal{B}$ implies $\Im\rho > \tau$;

(iii) For every $r \geqslant 2$ distinct elements $a_1, \dots, a_r$ of F_q^*, $\mathcal{B}$ is a barrier for the property $s(\{a_1, \dots, a_r\}) > r(r-1)$;

(iv) If $z_{\mathcal{B}}$ holds, then for every $r \geqslant 2$ distinct elements $a_1, \dots, a_r$ of F_q^*, $S_q(\{a_1, \dots, a_r\}) \leqslant r(r-1)$.

Proof. Suppose F_q^* is generated by $g_1, \dots, g_m$, which have orders $n_1, \dots, n_m$, where $n_1 n_2 \cdots n_m = \phi(q)$. Define χ_j by

$$\overline{\chi}_j(g_j) = e(1/n_j), \qquad \overline{\chi}_j(g_h) = 1 \quad (h \neq j).$$

Let γ be large depending on q, and

$$\tfrac{1}{2} < \beta_m < \beta_{m-1} < \cdots < \beta_1 < 1.$$

For $1 \leqslant j \leqslant m, 1 \leqslant k \leqslant 2$, let $n(\beta_j + ik\gamma, \chi_j^k) = c_{j,k}$. Also, for each j there is at least one k so that $n(\beta_j + ik\gamma, \chi_j^k) \geqslant 1$. In what follows, implied constants depend on q, γ and the numbers $c_{j,k}$. For each $a \in F_q^*$ write

$$a \equiv g_1^{\alpha_1(a)} \cdots g_m^{\alpha_m(a)} \pmod q, \quad 0 \leqslant \alpha_j(a) \leqslant n_j - 1.$$

Let $\mathcal{F}$ be β_m-similar to $\mathcal{P}_q$. By (1.8) and (1.9),

(5.1)
$$\begin{aligned}\Delta_{a,b}(u) &:= -\frac{u\phi(q)}{2}\left[F_{q,a}(e^u) - F_{q,b}(e^u)\right] \\ &= \sum_{j=1}^{m} e^{\beta_j u}\left(f_j(u, \alpha_j(a)) - f_j(u, \alpha_j(b))\right) + o(e^{\beta_m u}), \quad (u \to \infty)\end{aligned}$$

where

$$f_j(u, \alpha) = \Re \sum_{k=1}^{2} \frac{c_{j,k} e(k\alpha/n_j)}{\beta_j + ik\gamma}\left[e^{ik\gamma u} + u e^{-\beta_j u}\int_2^{e^u} \frac{v^{\beta_j + ik\gamma}}{v \log^2 v}\, dv\right]. \tag{5.2}$$

Let $J(a,b) = \{j : \alpha_j(a) \neq \alpha_j(b)\}$. Then

(5.3)
$$\begin{aligned}H_{a,b}(u) &:= \sum_{j \in J(a,b)} e^{\beta_j u}\left(f_j(u, \alpha_j(a)) - f_j(u, \alpha_j(b))\right) \\ &= \Delta_{a,b}(u) + o\left(e^{\beta_m u}\right) \quad (u \to \infty).\end{aligned}$$

Lastly, define the periodic functions

$$\begin{aligned} w_{j,\alpha}(u) &= \Re \sum_{k=1}^{2} c_{j,k} e(k\alpha/n_j) \frac{e^{ik\gamma u}}{\beta_j + ik\gamma} \\ &= \sum_{k=1}^{2} \frac{c_{j,k}}{\sqrt{k^2\gamma^2+\beta_j^2}} \sin\left(k\gamma u + \frac{2\pi k \alpha}{n_j} + \tan^{-1} \frac{\beta_j}{k\gamma} \right). \end{aligned}$$

By Lemma 1.1 (the asymptotic for $f(\rho)$) and (5.2),

$$f_j(u,\alpha) = w_{j,\alpha}(u) + O(1/u). \tag{5.4}$$

Similarly,

(5.5)

$$\begin{aligned} \frac{d}{du} f_j(u,\alpha) &= w'_{j,\alpha}(u) + \sum_{k=1}^{2} \frac{c_{j,k} e(\frac{k\alpha}{n_j})}{\beta_j + ik\gamma} \left[(1-\beta_j u) e^{-\beta_j u} \int_2^{e^u} \frac{v^{\beta_j + ik\gamma}}{v \log^2 v}\, dv + \frac{e^{ik\gamma u}}{u} \right] \\ &= w'_{j,\alpha}(u) + O(1/u). \end{aligned}$$

Each function $w_j(u,\alpha)$ is periodic in u with period $2\pi/\gamma$. We choose the numbers β_j and $c_{j,k}$ so that the functions $w_{j,\alpha}$ have several properties:

(A) For each j and each pair of distinct integers $\alpha_1, \alpha_2 \in [0, n_j - 1]$, the equation

$$w_{j,\alpha_1}(u) = w_{j,\alpha_2}(u)$$

has only two solutions in $[0, 2\pi/\gamma)$. Call them $\theta_v(j,\alpha_1,\alpha_2)$, $v = 1, 2$;

(B) All the numbers $\theta_v(j,\alpha_1,\alpha_2)$ are nonzero and distinct, that is

$$\theta_{v_1}(j_1,\alpha_1,\alpha_2) = \theta_{v_2}(j_2,\alpha_3,\alpha_4) \text{ implies } v_1 = v_2, j_1 = j_2, \{\alpha_1,\alpha_2\} = \{\alpha_3,\alpha_4\};$$

(C) For all j, v and distinct α_1, α_2, if $\theta = \theta_v(j,\alpha_1,\alpha_2)$ then $w'_{j,\alpha_1}(\theta) - w'_{j,\alpha_2}(\theta) \ne 0$;

(D) Let $1 \leqslant j' < j \leqslant m$, $v \in \{1,2\}$, distinct $\alpha_1, \alpha_2 \in [0, n_{j'} - 1]$. Suppose $\alpha_3, \alpha_4, \alpha_5, \alpha_6 \in [0, n_j - 1]$ with $(\alpha_3, \alpha_4) \ne (\alpha_5, \alpha_6)$ and not both $\alpha_3 = \alpha_4$ and $\alpha_5 = \alpha_6$. If $\theta = \theta_v(j', \alpha_1, \alpha_2)$, then

$$w_{j,\alpha_3}(\theta) - w_{j,\alpha_4}(\theta) - [w_{j,\alpha_5}(\theta) - w_{j,\alpha_6}(\theta)] \ne 0.$$

Note: some cases of (D) are redundant, being covered by property (B). For example, if $\alpha_3 = \alpha_4$ and $\alpha_5 \ne \alpha_6$, or if $\alpha_3 = \alpha_5$ and $\alpha_4 \ne \alpha_6$, or if $\alpha_3 = \alpha_6$ and $\alpha_4 = \alpha_5$.

For integral ℓ let $u_\ell = \frac{2\pi}{\gamma}\ell$. We claim the following hold for large ℓ (depending on $\mathcal{F}$). Throughout the remainder of this proof, $o(1)$ refers to a function of ℓ which tends to 0 as $\ell \to \infty$.

(i) At $u = u_\ell, u_{\ell+1}, \ldots$, the ordering of the functions $F_{q,a}(e^u)$ $(a \in F_q^*)$ is the same;

(ii) For distinct $a, b \in F_q^*$, the sign changes of $\Delta_{a,b}(u)$ on $[u_\ell, u_{\ell+1}]$ occur within two intervals $I_1(a,b)$ and $I_2(a,b)$. All $\phi(q)(\phi(q)-1)$ of these intervals are disjoint, and the sign of each function $\Delta_{c,d}(u)$ at the endpoints of $I_v(a,b)$ depends only on a, b, c, d and v.

Together, (i) and (ii) imply the theorem. Indeed, the possible orderings of $F_{q,a_i}(e^u)$ $(1 \leqslant i \leqslant r)$ are precisely the orderings occurring at the endpoints of the intervals $I_v(a_i, a_j)$. There are $r(r-1)$ such intervals, and the ordering remains constant between two such intervals, so there are at most $r(r-1)$ different orderings.

First we prove (i). Let $W(j,\alpha) = w_{j,\alpha}(0)$ for each j, α and let $L \geqslant \ell$. For each $a, b \in F_q^*$ let $j_0 = j_0(a,b) := \min\{j \in J(a,b)\}$. By (5.3) and (5.4),

$$\Delta_{a,b}(u_L) = \exp(\beta_{j_0} u_L)\left(W(j_0, \alpha_{j_0}(a)) - W(j_0, \alpha_{j_0}(b)) + o(1)\right).$$

By (B), $W(j_0, \alpha_{j_0}(a)) \neq W(j_0, \alpha_{j_0}(b))$ and so $\Delta_{a,b}(u_L)$ has constant sign for $L \geqslant \ell$.

Next we prove (ii). Throughout suppose $u_\ell \leqslant u \leqslant u_{\ell+1}$. For sufficiently small δ (depending only on the functions $w_{j,\alpha}$) let

$$M(a,b) = \left\{u \in [u_\ell, u_{\ell+1}] : |H_{a,b}(u)| \leqslant \delta e^{\beta_m u}\right\}.$$

By (5.3), $\Delta_{a,b}(u) = 0$ implies $u \in M(a,b)$. Let $j_0 = j_0(a,b)$, $\alpha_1 = \alpha_{j_0}(a)$ and $\alpha_2 = \alpha_{j_0}(b)$. By (5.3), for $u \in M(a,b)$,

$$|f_{j_0}(u, \alpha_1) - f_{j_0}(u, \alpha_2)| = o(1).$$

which by (5.4) implies that for any fixed $\eta > 0$, if ℓ is large enough,

$$|w_{j_0,\alpha_1}(u) - w_{j_0,\alpha_2}(u)| \leqslant \eta. \tag{5.6}$$

Let Y be the set of u satisfying (5.6). By (A) and (C), if δ and η are small enough then Y is the union of two short intervals K_1, K_2, where $\theta_v(j_0, \alpha_1, \alpha_2) \in K_v$ for $v = 1, 2$. By (C), for some $\varepsilon > 0$, $w'_{j_0,\alpha_1}(u) - w'_{j_0,\alpha_2}(u)$ has constant sign and is at least ε in magnitude on each interval K_v. By (5.3), (5.4) and (5.5), $H_{a,b}(u)$ is monotone on each of K_1 and K_2. Therefore, $I_1(a,b) := M(a,b) \cap K_1$ and $I_2(a,b) := M(a,b) \cap K_2$ are closed intervals. At the endpoints of $I_1(a,b)$ and $I_2(a,b)$, $|H_{a,b}(u)| = \delta e^{\beta_m u}$ and thus $\operatorname{sgn}\Delta_{a,b}(u) = \operatorname{sgn}H_{a,b}(u)$. For each v, the sign of $H'_{a,b}(u)$ on K_v thus determines the sign of $\Delta_{a,b}(u)$ at the endpoints of $I_v(a,b)$. This in turn depends only on the sign of $w'_{j_0,\alpha_1}(u) - w'_{j_0,\alpha_2}(u)$ on K_v, which does not depend on ℓ.

Next, suppose $u \in I_v(a,b)$ and $\{a,b\} \neq \{c,d\}$. Let $j_0 = j_0(a,b)$, $\alpha_1 = \alpha_{j_0}(a)$, $\alpha_2 = \alpha_{j_0}(b)$. Then

$$|u - \theta| = o(1), \quad \text{where } \theta = \theta_v(j_0, \alpha_1, \alpha_2). \tag{5.7}$$

Let $j_1 = j_0(c,d)$, $\alpha_3 = \alpha_{j_1}(c)$, $\alpha_4 = \alpha_{j_1}(d)$. By (5.3), (5.4), and (5.7),

$$\Delta_{c,d}(u) = e^{\beta_{j_1} u}[w_{j_1,\alpha_3}(\theta) - w_{j_1,\alpha_4}(\theta) + o(1)].$$

If $j_0 \ne j_1$ or $\{\alpha_1, \alpha_2\} \ne \{\alpha_3, \alpha_4\}$, $w_{j_1,\alpha_3}(\theta) - w_{j_1,\alpha_4}(\theta) \ne 0$ by (B), so $\Delta_{c,d}(u)$ has constant sign depending only on a, b, c, d, v. Next, suppose $j_0 = j_1$ and $\{\alpha_1, \alpha_2\} = \{\alpha_3, \alpha_4\}$. By swapping c and d if necessary, we may suppose that $\alpha_1 = \alpha_3$, $\alpha_2 = \alpha_4$. Let

$$j_2 = \min\{j \in J(a,b) \cup J(c,d) : (\alpha_j(a), \alpha_j(b)) \ne (\alpha_j(c), \alpha_j(d))\}.$$

Such j_2 exists because $\{a, b\} \ne \{c, d\}$. Also, by our assumptions on $j_0, \alpha_1, \dots, \alpha_4$, we have $j_2 > j_0$. By (5.3), (5.4), and (5.7),

$$\begin{aligned} \Delta_{c,d}(u) &= [\Delta_{c,d}(u) - \Delta_{a,b}(u)] + \Delta_{a,b}(u) \\ &= e^{\beta_{j_2} u}\big(w_{j_2}(\theta, \alpha_{j_2}(c)) - w_{j_2}(\theta, \alpha_{j_2}(d)) - [w_{j_2}(\theta, \alpha_{j_2}(a)) - w_{j_2}(\theta, \alpha_{j_2}(b))] \\ &\qquad + o(1)\big). \end{aligned}$$

By (D), the right side has constant sign, depending only on a, b, c, d, v. This completes the proof of (ii).

It remains to select numbers $\beta_1, \dots, \beta_m$ and $c_{j,k}$ so that (A)-(D) are satisfied. Write for short

$$z_j = \frac{\beta_j}{\gamma}, \quad \varepsilon_j = \tan^{-1} z_j, \quad \nu_j = \tan^{-1} \frac{z_j}{2}.$$

Let M be a large integer, depending only on q. We think of β_i and M as being fixed, while $\gamma \to \infty$. In what follows constants implied by O and $\ll$ will not depend on M or on γ.

If $n_j = 2$, take $c_{j,1} = 1$ and $c_{j,2} = 0$. In this case, we have

$$w_{j,\alpha}(u) = \frac{\sin(\gamma u + \pi\alpha + \varepsilon_j)}{\sqrt{\gamma^2 + \beta_j^2}}. \tag{5.8}$$

If $n_j \geqslant 4$, we take $c_{j,1} = M$, $c_{j,2} = 1$. In this case

$$w_{j,\alpha}(u) = \frac{M \sin(\gamma u + \frac{2\pi\alpha}{n_j} + \varepsilon_j)}{\sqrt{\gamma^2 + \beta_j^2}} + \frac{\sin(2\gamma u + \frac{4\pi\alpha}{n_j} + \nu_j)}{\sqrt{4\gamma^2 + \beta_j^2}}. \tag{5.9}$$

In particular, for fixed j, $w_{j,\alpha}(u) = w_{j,0}(u + \frac{2\pi\alpha}{\gamma n_j})$. Let $J_1 = \{j : n_j = 2\}$ and $J_2 = \{j : n_j \geqslant 4\}$. The functions $w_{j,0}(u)$ with $j \in J_1$ are very close to the function $\frac{1}{\gamma}\sin(\gamma u)$, and the functions $w_{j,0}(u)$ with $j \in J_2$ are all very close to the function $\frac{M}{\gamma}\sin(\gamma u) + \frac{1}{2\gamma}\sin(2\gamma u)$. It is important, however, that the actual functions $w_{j,0}$ ($j \in J_2$) are not odd nor are they a shift of an odd function.

Assume throughout that $0 \leqslant u < 2\pi/\gamma$. Consider first the equation

$$w_{j,\alpha_1}(u) = w_{j,\alpha_2}(u), \qquad \text{where } 0 \leqslant \alpha_1 < \alpha_2 \leqslant n_j - 1. \tag{5.10}$$

If $j \in J_1$ then $\alpha_1 = 0, \alpha_2 = 1$ and the solutions of (5.10) are

$$\gamma u \in \{\pi - \varepsilon_j, 2\pi - \varepsilon_j\}. \tag{5.11}$$

Since the numbers ε_j are distinct and $O(1/\gamma)$ in magnitude, all such solutions (for varying j) are distinct and non-zero. Similarly, when $j \in J_2$ and $\alpha_2 = \alpha_1 + \frac{1}{2}n_j$, (5.9) implies that the solutions of (5.10) are

$$\gamma u \in \{\pi(1 - 2\alpha_1/n_j) - \varepsilon_j, \pi(2 - 2\alpha_1/n_j) - \varepsilon_j\}. \tag{5.12}$$

Again these numbers are all distinct and non-zero (for varying j and α_1), and distinct from the numbers in (5.11). Finally, suppose $j \in J_2$ and $\alpha_2 - \alpha_1 \neq \frac{1}{2}n_j$. We make use of the following expression for $w_{j,\alpha}(u)$ which avoids square roots:

$$\gamma w_{j,\alpha}(u) = M\frac{z_j \cos(\omega) + \sin(\omega)}{1 + z_j^2} + \frac{z_j \cos(2\omega) + 2\sin(2\omega)}{4 + z_j^2}, \ \omega = \gamma u + \frac{2\pi\alpha}{n_j}. \tag{5.13}$$

Making the change of variables $y = \gamma u + \frac{\pi}{n_j}(\alpha_1 + \alpha_2)$, define

$$g(y) = g(y; j, \alpha_1, \alpha_2) = \frac{-\gamma}{2}(1 + z_j^2)(4 + z_j^2)(w_{j,\alpha_1}(u) - w_{j,\alpha_2}(u)).$$

Using some trigonometric identities with (5.13), we have

(5.14)
$$\begin{aligned} g(y) &= M(4 + z_j^2)\sin B(\cos y - z_j \sin y) + (1 + z_j^2)\sin 2B(2\cos 2y - z_j \sin 2y) \\ &= \sin B\left[M(4 + z_j^2)(\cos y - z_j \sin y) + 2(1 + z_j^2)\cos B(2\cos 2y - z_j \sin 2y)\right], \end{aligned}$$

where

$$B = \frac{\pi(\alpha_2 - \alpha_1)}{n_j} \in \left\{\frac{k\pi}{n_j} : 1 \leqslant k \leqslant n_j - 1, k \neq n_j/2\right\}. \tag{5.15}$$

Since $\sin B \neq 0$ by (5.15), combining (5.10) and (5.14) gives the approximation

$$4M\cos y + 4\cos 2y \cos B = O(M/\gamma). \tag{5.16}$$

We may assume $\gamma \geqslant M$. Thus $|\cos y| \ll 1/M$ and consequently $|\sin y| = 1 + O(1/M^2)$, $\cos 2y = -1 + O(1/M^2)$, and $|y \pm \pi/2| \ll 1/M$. For such y, $|g'(y)| \gg M$, so there are exactly two solutions of (5.10), one with y near $\pi/2$ and the other with y near $-\pi/2$. This proves (A). When $u = 0$, i.e. $y = \frac{\pi}{n_j}(\alpha_1 + \alpha_2)$, (5.15) implies

$|g(y)| \gg M$ unless $\alpha_1 + \alpha_2 \in \{n_2/2, 3n_j/2\}$. In this case $g(y) = -4\cos B \sin B + O(M/\gamma) \neq 0$ by (5.15). This proves that every $\theta_v(j, \alpha_1, \alpha_2) \neq 0$.

For the second part of (B), consider the equation

$$\theta_{v_1}(j_1, \alpha_1, \alpha_2) = \theta_{v_2}(j_2, \alpha_3, \alpha_4).$$

This implies that for some u,

$$w_{j_1,\alpha_1}(u) = w_{j_1,\alpha_2}(u), \quad w_{j_2,\alpha_3}(u) = w_{j_2,\alpha_4}(u). \tag{5.17}$$

We cannot have $j_1 = j_2 \in J_1$. First suppose $j_1, j_2 \in J_2$. We may assume $\alpha_1 < \alpha_2$, $\alpha_3 < \alpha_4$ and either $j_1 \neq j_2$ or $\{\alpha_1, \alpha_2\} \neq \{\alpha_3, \alpha_4\}$. If $j_1 = j_2 = j$ and $\alpha_i = \alpha_k$ for some $i \neq k$, then $w_{j,\alpha_1}(u) = w_{j,\alpha_2}(u) = w_{j,\alpha_3}(u) = w_{j,\alpha_4}(u)$, the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ contains three distinct elements, and the function $w_{j,0}$ takes some value three times, which is impossible.

By (5.14), we have the system of equations

(5.18)
$$\begin{aligned} &M(4 + z_{j_1}^2)(\cos y_1 - z_{j_1} \sin y_1) + 2(\cos B_1)(1 + z_{j_1}^2)(2\cos 2y_1 - z_{j_1} \sin 2y_1) = 0, \\ &M(4 + z_{j_2}^2)(\cos y_2 - z_{j_2} \sin y_2) + 2(\cos B_2)(1 + z_{j_2}^2)(2\cos 2y_2 - z_{j_2} \sin 2y_2) = 0, \end{aligned}$$

where

$$y_1 = \gamma u + \frac{\pi(\alpha_1 + \alpha_2)}{n_{j_1}}, y_2 = \gamma u + \frac{\pi(\alpha_3 + \alpha_4)}{n_{j_2}}, B_1 = \frac{\pi(\alpha_2 - \alpha_1)}{n_{j_1}}, B_2 = \frac{\pi(\alpha_4 - \alpha_3)}{n_{j_2}}.$$

As before, $|\cos y_k| = O(1/M)$ for $k = 1, 2$. Since $y_1 - y_2$ is an integral multiple of $\pi/\phi(q)$ and γ is large, $\cos y_1 = \pm \cos y_2$. As a consequence, $\cos 2y_1 = \cos 2y_2 = -1 + O(1/M^2)$ and so by (5.18),

$$4M \cos y_1 + 4\cos 2y_1 \cos B_k = O(M/\gamma) \qquad (k = 1, 2).$$

This in turn implies that either $\cos y_1 = \cos y_2$ and $B_1 = B_2$ or that $\cos y_1 = -\cos y_2$ and $B_1 = \pi - B_2$. Consider four cases: (i) $\cos y_1 = \cos y_2$, $\sin y_1 = \sin y_2$, (ii) $\cos y_1 = \cos y_2$, $\sin y_1 = -\sin y_2$,
(iii) $\cos y_1 = -\cos y_2$, $\sin y_1 = \sin y_2$, (iv) $\cos y_1 = -\cos y_2$, $\sin y_1 = -\sin y_2$. In cases (ii) and (iii), subtracting or adding the two equations in (5.18) yields

$$(z_{j_1} + z_{j_2})(4M \sin y_1 - 2\cos B_1 \sin 2y_1) = O(M/\gamma^2),$$

which is not possible given that $|\sin y_1| = 1 + O(1/M^2)$. In case (i) $y_1 = y_2$ and, together with $B_1 = B_2$, implies that $j_1 \neq j_2$ and hence $z_{j_1} \neq z_{j_2}$. In case (iv) $|y_1 - y_2| = \pi$ and, together with $B_1 = \pi - B_2$ implies that two of the numbers $\frac{\alpha_1}{n_{j_1}}$, $\frac{\alpha_2}{n_{j_1}}$, $\frac{\alpha_3}{n_{j_2}}$, $\frac{\alpha_4}{n_{j_2}}$ are equal, therefore, $j_1 \neq j_2$ and $z_{j_1} \neq z_{j_2}$.

Again subtracting the two equations in (5.18) in case (i) and adding in case (iv) produces

$$(z_{j_1} - z_{j_2})(4M \sin y_1 - 2\cos B_1 \sin 2y_1) = O(M/\gamma^2),$$

which likewise gives a contradiction. Therefore, (5.17) is impossible when $j_1, j_2 \in J_2$.

If $j_1 \in J_1$, $j_2 \in J_2$, then by (5.11), $\gamma u \in \{\pi - \varepsilon_{j_1}, 2\pi - \varepsilon_{j_1}\}$. We have seen that (5.17) is impossible in the case $\alpha_4 = \alpha_3 + \frac{1}{2}n_{j_2}$. Assume that the last equality does not hold and define $y = \gamma u + \pi(\alpha_3+\alpha_4)/n_{j_2}$, $B = \pi(\alpha_4 - \alpha_3)/n_{j_2}$ By (5.17), $g(y; j_2, \alpha_3, \alpha_4) = 0$. From (5.14), $|\cos y| \ll 1/M$, and thus $\pi(\alpha_3+\alpha_4)/n_{j_2} \in \{\pi/2, 3\pi/2\}$. This implies the stronger inequality $|\cos y| = |\sin \varepsilon_{j_1}| \leqslant 1/\gamma$ and as a consequence $\cos 2y \leqslant -1 + 2/\gamma^2$. Applying (5.14) again we see that $4\cos B \cos 2y = O(M/\gamma)$, which by (5.15) is impossible. This completes the proof of (B).

Next we verify (C). If $j \in J_1$, then $\alpha_1 = 0$, $\alpha_2 = 1$, θ satisfies $\sin(\gamma\theta + \varepsilon_j) = 0$ and thus $\cos(\gamma\theta + \varepsilon_j) = \pm 1$, so $w'_{j,0}(\theta) \neq w'_{j,1}(\theta)$. If $j \in J_2$ and $|\alpha_1 - \alpha_2| = \frac{1}{2}n_j$, the situation is the same as with $j \in J_1$ by (5.9). Assume $0 \leqslant \alpha_1 < \alpha_2 < n_j$ and $\alpha_2 - \alpha_1 \neq \frac{1}{2}n_j$. Define B and y as in (5.14), (5.15). Suppose u satisfies (5.10) and also the equation $w'_{j,\alpha_1}(u) = w'_{j,\alpha_2}(u)$. Differentiating (5.14) gives

$$\sin y(4M + 16\cos y \cos B) = O(M/\gamma),$$

which is impossible since $|\sin y| = 1 + O(1/M^2)$. Thus condition (C) is verified.

We verify condition (D) indirectly. Condition (B) covers the situation when $\alpha_3 = \alpha_4$, $\alpha_5 = \alpha_6$, $\alpha_3 = \alpha_5$, $\alpha_4 = \alpha_6$ or there are at most two distinct values among $\alpha_3, \dots, \alpha_6$. Henceforth assume none of these conditions occurs and, moreover, $\alpha_3 < \alpha_4$, $\alpha_5 < \alpha_6$. Fix $j', v, \alpha_1, \alpha_2$ and put $u = \theta_v(j', \alpha_1, \alpha_2)$. Fix $j, \alpha_3, \dots, \alpha_6$ and define

$$y_1 = \gamma u + \frac{\pi(\alpha_3+\alpha_4)}{n_j}, y_2 = \gamma u + \frac{\pi(\alpha_5+\alpha_6)}{n_j}, B_1 = \frac{\pi(\alpha_4-\alpha_3)}{n_j}, B_2 = \frac{\pi(\alpha_6-\alpha_5)}{n_j}.$$

Using (5.14), the equation in (D) becomes

$$\begin{aligned}(5.19)\quad P(z_j) :&= M(4+z_j^2)\left[\sin B_1(\cos y_1 - z_j \sin y_1) - \sin B_2(\cos y_2 - z_j \sin y_2)\right] \\ &+ (1+z_j^2)\left[\sin 2B_1(2\cos 2y_1 - z_j \sin 2y_1) - \sin 2B_2(2\cos 2y_2 - z_j \sin 2y_2)\right] = 0.\end{aligned}$$

We shall prove that for large M the polynomial P is not identically zero. The conclusion is that given $\beta_1, \dots, \beta_{j-1}$, there are a finite number of β_j which would lead to failure of (D). Consequently, we can always choose an admissible β_j from within a short interval.

Note that we have proved (by (5.14)) that $\gamma u = \frac{\pi\alpha}{n_{j'}} + O(1/M)$ for some integer α. Therefore, there are $\tilde{y}_1 = \frac{\pi l_1}{\phi(q)}$ and $\tilde{y}_2 = \frac{\pi l_2}{\phi(q)}$ with some integers l_1 and l_2 such that $y_1 = \tilde{y}_1 + O(1/M)$, $y_2 = \tilde{y}_2 + O(1/M)$.

Suppose that all the coefficients of P are zero. The constant term is $4aM + O(1)$, $a = \sin B_1 \cos \tilde{y}_1 - \sin B_2 \cos \tilde{y}_2$. Since a can take finitely many values, we conclude from $4aM + O(1) = 0$ that

$$a = \sin B_1 \cos \tilde{y}_1 - \sin B_2 \cos \tilde{y}_2 = 0. \tag{5.20}$$

In the same way, considering the coefficients of z_j, we get

$$\sin B_1 \sin \tilde{y}_1 - \sin B_2 \sin \tilde{y}_2 = 0. \tag{5.21}$$

Taking into account that $\sin B_1 > 0$, $\sin B_2 > 0$, we deduce from (5.20) and (5.21) that

$$\sin B_1 = \sin B_2, \quad \tilde{y}_1 = \tilde{y}_2. \tag{5.22}$$

Further, the last equality implies that in fact $y_1 = y_2$. This in turn implies that the sums of terms containing M in the coefficients of P are zero. Therefore, the conditions that the constant term and the coefficient of z_j in P are zero mean that

$$\cos 2y_1(\cos B_1 - \cos B_2) = 0, \quad \sin 2y_1(\cos B_1 - \cos B_2) = 0.$$

It follows from these equalities and (5.22) that

$$B_1 = B_2. \tag{5.23}$$

Finally, from (5.22) and (5.23) we obtain $\alpha_3 = \alpha_5$ and $\alpha_4 = \alpha_6$, which does not agree with our assumptions and completes the proof of Theorem 5.1. □

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