GEOMETRIC PROPERTIES OF POINTS ON MODULAR HYPERBOLAS

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ABSTRACT. Given an integer $n \geqslant 2$, let $\mathcal{H}_n$ be the set

$$\mathcal{H}_n = \{(a,b) \ : \ ab \equiv 1 \pmod n, \ 1 \leqslant a, b \leqslant n-1\}$$

and let $M(n)$ be the maximal difference of $b-a$ for $(a,b) \in \mathcal{H}_n$. We prove that for almost all n, $n - M(n) = O\left(n^{1/2+o(1)}\right)$. We also improve some previously known upper and lower bounds on the number of vertices of the convex closure of $\mathcal{H}_n$.

1. INTRODUCTION

This paper pursues two goals. We prove a weak version of a conjecture in the paper [4] and improve some results in [9]. To put our results in context, we begin by discussing the contents of [4] and [9].

For an integer $n \geqslant 2$, we define the modular hyperbola, $\mathcal{H}_n$, to be the set

$$\mathcal{H}_n = \{(a,b) \ : \ ab \equiv 1 \pmod n, \ 1 \leqslant a, b \leqslant n-1\}.$$

There are many interesting and productive questions one can pose about this set. One is the study of $M(n)$, the maximal difference between the components of points of $\mathcal{H}_n$, that is,

$$M(n) = \max\{b - a \ : \ (a,b) \in \mathcal{H}_n\}.$$

This function has been studied in two papers [8, 4]. In [8, Theorem 4] it is proved via Kloosterman sums that $n - M(n) \leqslant n^{3/4+o(1)}$, and in [4] it is shown that for almost all n

$$n - M(n) \geqslant n^{1/2}(\log n)^{\delta/2}(\log\log n)^{3/4} f(n),$$

where

$$\delta = 1 - \frac{1 + \log\log 2}{\log 2} = 0.08607\ldots,$$

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and $f(n)$ is an arbitrary function with $\lim_{n\to\infty} f(n) = 0$. Furthermore, in [4], the authors have conjectured that if $g(n) \to \infty$ as $n \to \infty$, then

$$n - M(n) \leqslant n^{1/2}(\log n)^{\delta/2}(\log\log n)^{3/4} g(n)$$

for almost all n, and have given a heuristic for this statement. We prove a weaker form of this conjecture.

Theorem 1. *For every $\varepsilon > 0$ and $A > 0$, we have $n - M(n) = O(n^{1/2+\varepsilon})$ for all integers $n \leqslant x$ with at most $O(x/(\log x)^A)$ exceptions.*

In particular, we see that $n - M(n) = n^{1/2+o(1)}$ for almost all n. After proving Theorem 1, we turn our attention to improving certain results that have appeared in [9]. Following [9], let $\mathcal{C}_n$ denote the convex closure of the set $\mathcal{H}_n$ and let $v(n)$ denote the number of vertices of $\mathcal{C}_n$. The paper [9] is an attempt to determine asymptotic bounds for $v(n)$, and in this the authors have only been partly successful. Let us describe some elementary properties of $\mathcal{H}_n$ and $\mathcal{C}_n$.

The first is that the lines $y = x$ and $y = n - x$ are lines of symmetry of $\mathcal{H}_n$. These symmetries reduce the amount of work needed to determine the vertices of $\mathcal{C}_n$, as one can restrict the search to the vertices of $\mathcal{C}_n$ that lie in the triangle $\mathcal{T}_n$ with vertices $(0, 0)$, $(0, n)$ and $(n/2, n/2)$. Following [9], let $(a_0, b_0) = (1, 1), (a_1, b_1), \ldots, (a_s, b_s)$, with $a_0 < a_1 < \ldots < a_s$, be the vertices of C_n in $\mathcal{T}_n$. Then $M(n) = b_s - a_s$, that is, the maximum difference is achieved by the highest vertex of $\mathcal{C}_n$ in $\mathcal{T}_n$.

We illustrate this with the graph below of $\mathcal{H}_{47}$ with the lines of symmetry $y = x$ and $y = 47 - x$. We note that $(a_1, b_1) = (2, 24)$ and $(a_s, b_s) = (a_2, b_2) = (10, 33)$.

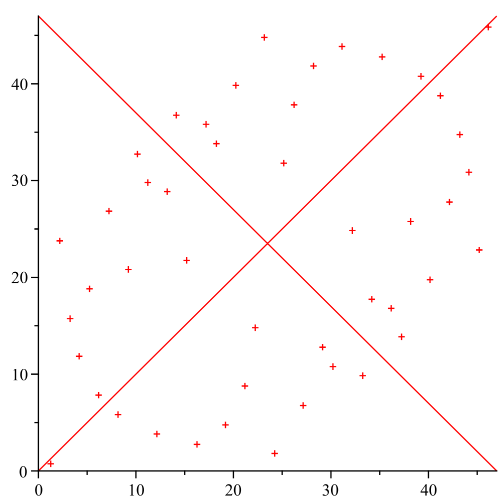

Figure 1. The curve $\mathcal{H}_{47}$ with the lines of symmetry $y = x, y + x = 47$

One of the first results in [9] is that for all $n > 1$,

$$v(n) \geqslant 2(\tau(n-1)-1)$$

where $\tau(k)$ is the number of positive integer divisors of k. The proof follows from observing that the lattice points on the curves

$$x(n-y) = n-1 \text{ and } (n-x)y = n-1, \text{ with } 1 \leqslant x, y \leqslant n-1,$$

belong to $\mathcal{C}_n$ with the points $(1,1)$ and $(n-1, n-1)$ being common to both curves. We illustrate this in the graph below.

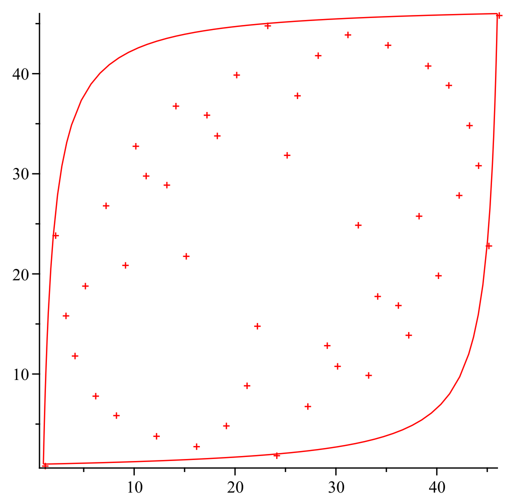

Figure 2. The curves $x(47-y) = 46$, $(47-x)y = 46$ enclosing $\mathcal{H}_{47}$

This estimate is tight as $v(n) = 2(\tau(n-1)-1)$ for infinitely many integers n. Specifically in [9, Theorem 3.2] it is shown that

$$\#\{n \leqslant x \; : \; v(n) = 2(\tau(n-1)-1)\} \gg \frac{x}{\log x},$$

where, as usual, the notations $U \ll V$ and $V \gg U$ are equivalent to $U = O(V)$, (throughout the paper, the implied constants may depend on the positive parameters ε and B, and are absolute otherwise).

The authors [9, Theorem 3.4 (b)] then give a conditional proof of $v(n) > 2(\tau(n-1)-1)$ for almost all n under the hypothesis that for almost all n, $n - M(n) \leqslant n^{1/2+o(1)}$. The proof is by combining a result of [3] with the inequality $n - M(n) \leqslant n^{1/2+o(1)}$ to obtain that for almost all n, the vertex (a_s, b_s) does not lie on the curve $x(n-y) = n-1$. Hence, by proving Theorem 1 we obtain the following unconditional result.

Corollary 2. *The set of integers n for which $v(n) > 2(\tau(n-1)-1)$ has asymptotic density 1.*

Another result of [9] is that $v(n)/\tau(n-1) \neq O(1)$. Specifically it is shown in [9] that for infinitely many primes p,

$$v(p+1) \geqslant \exp\left(\left(\frac{2\log 2}{11} + o(1)\right)\frac{\log p}{\log\log p}\right). \tag{1.1}$$

The basic idea of the proof is to find primes, p, such that $2p+1$ has "many" factors. This is achieved by combining the prime number theorem with the Heath-Brown estimate [7] on the smallest prime in an arithmetic progression (see [9, Theorem 3.5]). In this paper we improve (1.1) by applying a result of Alford, Granville and Pomerance [1, Theorem 2.1] on the distribution of primes in almost all arithmetic progressions.

Theorem 3. *There are infinitely many primes p with*

$$v(p+1) \geqslant \exp\left(\left(\frac{5\log 2}{12} + o(1)\right)\frac{\log p}{\log\log p}\right).$$

The set of vertices of $\mathcal{C}_n$ seems to be a "hybrid" set in the sense that Tao uses it in [12, page 156]. The structured part of this set are the vertices that arise from the divisors of $n-1$. The remaining vertices seem to arise from a combination of pseudorandomness and the structure of divisors of $nj-1$ for some "small" values of $j \geqslant 2$. A recurrent theme in our attempts to handle the difficulties arising from the "pseudorandomness" of $v(n)$ is to apply the properties of the special vertex (a_s, b_s). So for example the bound $b_s - a_s = n - M(n) \leqslant n^{3/4+o(1)}$ immediately gives us that

$$v(n) \leqslant n^{3/4+o(1)}. \tag{1.2}$$

Unfortunately this is a pretty crude bound, as the numerics in [9] indicate that $v(n) \leqslant n^{o(1)}$. (We should mention that in [9, Section 5.2] there are a couple of "reasonable" numerical approximations to the difference $v(n) - 2(\tau(n-1)-1)$, but these are just guesses.) In this paper we make a small improvement to (1.2) by using a result of Andrews [2] on the number of integral vertices of convex flat (that is, 2-dimensional) polygons. We prove the following result.

Theorem 4. *We have*

$$v(n) \leqslant n^{7/12+o(1)}.$$

2. Preliminaries

We need the following special case of [5, Proposition 1].

Lemma 2.1. *Let L, N and Q be arbitrary real numbers, which for a fixed $\varepsilon > 0$ satisfy the inequalities*

$$2 \leqslant L^{\varepsilon} \leqslant N \leqslant L^{1/2-\varepsilon} \qquad \textit{and} \qquad 2 \leqslant Q \leqslant L^{3/4-\varepsilon},$$

and let $(\alpha_m)_{m\in[L,2L]}$ *be an arbitrary sequence of complex numbers with* $|\alpha_m|\leqslant 1$. *Then, for every fixed* $A>0$ *we have*

$$\sum_{1\leqslant q\leqslant Q}\left(\sum_{\substack{L<m\leqslant 2L\\ N<n\leqslant 2N\\ mn\equiv 1 \pmod q}}\alpha_m-\frac{1}{\varphi(q)}\sum_{\substack{L<m\leqslant 2L\\ N<n\leqslant 2N\\ \gcd(mn,q)=1}}\alpha_m\right)\ll LN(\log L)^{-A}.$$

Let $\varphi(x;n)=\#\{a\ :\ 1\leqslant a\leqslant x,\ \gcd(a,n)=1\}$ be the standard extension of the Euler function. Then, by the inclusion-exclusion principle, we have

$$\varphi(x;n)=\sum_{d|n}\left[\frac{x}{d}\right]\mu(d),$$

where $\mu(d)$ is the Möbius function. We need the following two consequences of this identity.

Lemma 2.2. *Let* $I,L\in\mathbb{Z}^+$*; let* $x\geqslant 0$*; and let* $\tau^*(L)$ *denote the number of square-free divisors of* L*. Then,*

$$\sum_{\substack{I<j\leqslant I+J\\ \gcd(j,L)=1}}1=\frac{\varphi(L)}{L}J+O(\tau^*(L)).$$

and

$$\sum_{\substack{I<j\leqslant I+J\\ \gcd(j,L)=1}}\frac{1}{j}=\frac{\varphi(L)}{L}\log(1+J/I)+O\left(\tau^*(L)/I\right).$$

We remark that when we apply Lemma 2.2 we replace $\tau^*(L)$ in the error term with $L^{o(1)}$. Finally, we recall the following special case of a general result of Andrews [2].

Lemma 2.3. *A convex 2-dimensional polygon of area* S*, with all vertices on the lattice* $\mathbb{Z}^2$*, has at most* $O(S^{1/3})$ *vertices.*

3. Proof of Theorem 1

Let m be a positive integer, and let Q and R be two positive real numbers. We define $\mathcal{V}(m;Q,R)$ to be the set

$$\left\{(q,r)\in\mathbb{Z}^2\ :\ \frac{Q}{2}<q\leqslant Q,\frac{mR+1}{q}<r\leqslant\frac{2mR+1}{q},\gcd(qr,m)=1\right\}.$$

This set plays a central role in our proof and we require the following asymptotic for $\#\mathcal{V}(m;Q,R)$:

Lemma 3.1. *We have,*

$$\#\mathcal{V}(m;Q,R) = \frac{\varphi(m)^2}{m} R\log 2 + O\left(Qm^{o(1)}\right).$$

Proof.

$$\begin{aligned}\#\mathcal{V}(m;Q,R) &= \sum_{\substack{Q/2<q\leqslant Q\\ \gcd(q,m)=1}} \sum_{\substack{(mR+1)/q<r\leqslant(2mR+1)/q\\ \gcd(r,m)=1}} 1\\ &= \sum_{\substack{Q/2<q\leqslant Q\\ \gcd(q,m)=1}} \left(\frac{\varphi(m)R}{q} + O\left(m^{o(1)}\right)\right)\\ &= R\varphi(m) \sum_{\substack{Q/2<q\leqslant Q\\ \gcd(q,m)=1}} \frac{1}{q} + O\left(m^{o(1)}Q\right).\end{aligned}$$

Applying Lemma 2.2 we conclude the proof. □

We are now ready to prove Theorem 1. Let m be a positive integer; let Q and R be two positive real numbers; and let $N(m;Q,R)$ denote the number of solutions to the congruence:

$$qr \equiv -1 \pmod m, \qquad (q,r) \in \mathcal{V}(m;Q,R).$$

If this congruence has a solution, then $M(m) \geqslant m-r-q$, that is, $r+q \geqslant m-M(m)$. So the plan to prove the result is to find appropriate bounds for Q and R, and then apply Lemma 2.1 to obtain $r+q \leqslant L^{1/2+o(1)}$ for $L \leqslant m$.

For $L > Q \geqslant 2$, with $L < m \leqslant 2L$, we consider the sum

$$\begin{aligned}W(L;Q,R) &= \sum_{L<m\leqslant 2L} \left| N(m;Q,R) - \frac{1}{\varphi(m)}\#\mathcal{V}(m;Q,R)\right|\\ &= \sum_{L<m\leqslant 2L} \alpha_m \left(N(m;Q,R) - \frac{1}{\varphi(m)}\#\mathcal{V}(m;Q,R)\right)\\ &= \sum_{L<m\leqslant 2L} \alpha_m \left(N(m;Q,R) - \frac{1}{\varphi(m)} \sum_{(q,r)\in\mathcal{V}(m;Q,R)} 1\right)\\ &= U_1 - U_2,\end{aligned} \tag{3.1}$$

where $\alpha_m = \pm 1$,

$$U_1 = \sum_{Q/2<q\leqslant Q} \sum_{\substack{L<m\leqslant 2L\\ \gcd(m,q)=1}} \alpha_m \sum_{\substack{(mR+1)/q<r\leqslant(2mR+1)/q\\ rq\equiv -1 \pmod m}} 1,$$

$$U_2 = \sum_{Q/2<q\leqslant Q} \sum_{\substack{L<m\leqslant 2L\\ \gcd(m,q)=1}} \frac{\alpha_m}{\varphi(m)} \sum_{\substack{(mR+1)/q<r\leqslant(2mR+1)/q\\ \gcd(r,m)=1}} 1.$$

We now replace the condition $rq \equiv -1 \pmod m$ with the equation $rq = mn - 1$, where for $(r,q) \in \mathcal{V}(m;Q,R)$ we have $R < n \leqslant 2R$.

Therefore,

$$U_1 = \sum_{Q/2<q\leqslant Q} \sum_{\substack{L<m\leqslant 2L\\ \gcd(m,q)=1}} \alpha_m \sum_{\substack{R<n\leqslant 2R\\ mn\equiv 1 \pmod q}} 1.$$

We now fix some $\varepsilon > 0$ and take

$$Q = L^{1/2+\varepsilon} \qquad \text{and} \qquad R = L^{\varepsilon}. \tag{3.2}$$

Then Lemma 2.1 can be applied (with q varying from $Q/2$ to Q), followed by an application of Lemma 2.2. We obtain

$$\begin{aligned} U_1 &= \sum_{Q/2<q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{L<m\leqslant 2L\\ R<n\leqslant 2R\\ \gcd(mn,q)=1}} \alpha_m + O\left(LR(\log L)^{-(A+\epsilon/2)}\right) \\ &= R \sum_{Q/2<q\leqslant Q} \frac{1}{q} \sum_{\substack{L<m\leqslant 2L\\ \gcd(m,q)=1}} \alpha_m + O\left(LR(\log L)^{-(A+\epsilon/2)}\right). \end{aligned}$$

Again by Lemma 2.2, we have

$$\begin{aligned} U_2 &= \sum_{Q/2<q\leqslant Q} \sum_{\substack{L<m\leqslant 2L\\ \gcd(m,q)=1}} \frac{\alpha_m}{\varphi(m)} \left(\frac{\varphi(m)R}{q} + O\left(L^{\epsilon/4}\right)\right) \\ &= R \sum_{Q/2<q\leqslant Q} \frac{1}{q} \sum_{\substack{L<m\leqslant 2L\\ \gcd(m,q)=1}} \alpha_m + O\left(L^{1+\epsilon/4}\right). \end{aligned} \tag{3.3}$$

Inserting the bounds for U_1 and U_2 into (3.1), we obtain

$$W(L;Q,R) \ll LR(\log L)^{-(A+\epsilon/2)}. \tag{3.4}$$

Combining Lemma 3.1 with (3.4) we get

$$\sum_{L<m\leqslant 2L} \left| N(m;Q,R) - \frac{\varphi(m)}{m} R \log 2 \right| \ll LR(\log L)^{-(A+\epsilon/2)}.$$

Since $\varphi(m) \gg m/\log\log m$, this shows that $N(m;Q,R) \geqslant 1$ for all $m \in (L, 2L]$ with at most

$$O\left(\frac{L\log\log L}{(\log L)^{A+\epsilon/2}}\right) \ll \frac{L}{(\log L)^A}$$

exceptions.

If $N(m;Q,R) \geqslant 1$ then we have a lattice point $(q,r) \in \mathcal{V}(m;Q,R)$ satisfying the congruence $qr \equiv -1 \pmod m$. We now get that

$$m - M(m) \leqslant r + q \ll L^{1/2+\varepsilon} \ll m^{1/2+\varepsilon}.$$

4. Proof of Theorem 3

Let p be a prime. A simple geometric calculation shows that every divisor d of $2p+1$, with $3 < d < (2p+1)/3$, gives rise to a lattice point on the curve $x(n-y) = 2p+1$ that is a vertex of $\mathcal{C}_{p+1}$. This immediately leads to the inequality

$$v(p+1) \geqslant 2(\tau(2p+1) - 3). \tag{4.1}$$

(See the beginning of the proof of [9, Theorem 3.5] for the details.) So the main difficulty is to show the existence of primes such that $\tau(2p+1)$ is large. This we do by applying the result of Alford, Granville and Pomerance [1, Theorem 2.1]. The next couple of paragraphs is devoted to setting up the hypotheses so that we can invoke this result.

We start by fixing an arbitrary $A > 12/5$ and a sufficiently small $\delta > 0$. We now consider the set $\mathcal{D}_{1/2,\delta}(x)$ as defined in [1, Theorem 2.1] (that is, we apply it with $\varepsilon = 1/2$, but we can choose any ε such that $0 < \varepsilon < 1$). Two parameters associated with $\mathcal{D}_{1/2,\delta}$ are the positive integer $D_{1/2,\delta}$ and the positive real number $x_{\varepsilon,\delta}$. We assume that $x \geqslant x_{1/2,\delta}$ is sufficiently large. We now need to determine a modulus q that satisfies three conditions:

- $q \leqslant x^{1/A-\delta}$;
- q has many prime factors;
- q is relatively prime to every element in $\mathcal{D}_{1/2,\delta}(x)$.

Let

$$\theta(x) = \sum_{\ell \leqslant x, \ell \text{ prime}} \log \ell$$

denote the Chebyshev function and let L be the largest integer that satisfies the inequality

$$\theta(L) - \log 2 \leqslant (1/A - \delta)\log x.$$

By the prime number theorem

$$L = \left(\frac{1}{A} - \delta + o(1)\right)\log x. \tag{4.2}$$

Let
$$D(x) = \prod_{d \in \mathcal{D}_{1/2,\delta}(x)} d, \qquad Q = \exp(\theta(L) - \log 2).$$
We now set q to be the integer
$$q = \frac{Q}{\gcd(Q, D(x))}.$$
Since $\#\mathcal{D}_{1/2,\delta}(x) \leqslant D_{1/2,\delta}$, we have
$$\tau(q) \geqslant 2^{\pi(L) - D_{1/2,\delta}} = 2^{\pi(L)+O(1)} = 2^{(1+o(1))L/\log L}, \tag{4.3}$$
and so we see that q indeed satisfies all three conditions that we listed.

On applying the bound of [1, Theorem 2.1] with $d = q$ and $y = x$, we see that for a sufficiently large x (depending only on A and δ) there is a prime $p \leqslant x$ in the arithmetic progression $2p \equiv -1 \pmod q$. Combining (4.2), (4.3) and the inequality $\tau(2p+1) \geqslant \tau(q)$ we obtain that
$$\tau(2p+1) \geqslant \exp\left(\left(\left(\frac{1}{A} - \delta\right)\log 2 + o(1)\right)\frac{\log x}{\log\log x}\right).$$
Using (4.1) and recalling that $A \geqslant 12/5$ and $\delta > 0$ are arbitrary, we conclude the proof of Theorem 3.

5. Proof of Theorem 4

We remind the reader that $(a_0, b_0), (a_1, b_1), \ldots, (a_s, b_s)$ denote the vertices of $\mathcal{C}_n$ that lie in the triangle with vertices $(0,0), (0,n)$ and $(n/2, n/2)$. Let C be the convex closure of the points $(a_0, b_0), (a_1, b_1), \ldots, (a_s, b_s)$. Then clearly C lies inside the rectangle with vertices $(1,1), (a_s, 1), (1, b_s)$ and (a_s, b_s), and consequently the area of C is at most $a_s \cdot b_s \leqslant n^{7/4+o(1)}$. We now invoke Lemma 2.3 to conclude that $s \leqslant n^{7/12+o(1)}$.

6. Comments

We note that one can also combine the arguments of the proof of Theorems 1 and 4 and to show that for almost all n we have
$$v(n) \leqslant n^{1/2+o(1)}.$$
Furthermore, it is easy to see that the proof of Theorem 4 generalizes to the number of vertices, $v_h(n)$, of the convex closure $\mathcal{C}_{h,n}$ of the hyperbola
$$\mathcal{H}_{h,n} = \{(a,b) \ : \ ab \equiv h \pmod n, \ 1 \leqslant x, y \leqslant n-1\}$$
for an arbitrary integer h satisfying $\gcd(h,n) = 1$. In particular, we have a full analogue of Theorem 4 for $v_h(n)$. Moreover, using [11, Theorem 1] one can easily derive that
$$v_h(n) = n^{1/2+o(1)}$$

for all but $o(\varphi(n))$ integers h with $1 \leqslant h \leqslant n-1$ and $\gcd(h,n) = 1$, where $\varphi(n)$ denotes the Euler function. Unfortunately, the result of Andrews [2] does not help in this case.

One can also use [10, Theorems 8 and 9] in conjunction with similar arguments to obtain results for the number of vertices of the convex closure of a multidimensional hyperbola. We recall that the result of Andrews [2] generalises to multidimensional polygons. Interestingly, Theorem 3 does not immediately generalise to $v_a(n)$ or the multidimensional case. Finally, we remark that the result of Harman [6] may possibly lead to a further improvement of Theorem 3.

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