Cycling Via Common Divisors

11019 [2003, 542]. *Proposed by Bernardo Recamán Santos, Universidad Sergio Arboleda, Bogotá, Colombia.*
(a) Find an integer N so that there is a block B of N consecutive integers that can be arranged cyclically so that adjacent pairs have a nontrivial common divisor.
(b)* Show that this can be done for all sufficiently large N.
Solution to (a) by Michail Reid, University of Central Florida, Orlando, FL. The smallest length of such a block is 47, and the first such block of length 47 begins at 29056075343306. For $0 \le i < 47$, define $a_i = 29056075343306 + b_i$, where $b_i = 0$, 11, 22, 33, 44, 27, 10, 3, 17, 31, 38, 45, 24, 39, 34, 29, 19, 14, 9, 4, 35, 6, 28, 41, 15, 2, 21, 40, 43, 37, 25, 13, 7, 1, 16, 36, 32, 30, 26, 20, 18, 12, 8, 42, 5, 46, 23 for $i = 0, 1, 2, \ldots, 46$, respectively. We check that $\gcd(a_i, a_{i+1}) = 11$, 11, 11, 11, 17, 17, 7, 7, 7, 7, 7, 7, 5, 5, 5, 5, 5, 5, 5, 31, 29, 2, 13, 13, 13, 19, 19, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 4, 2, 2, 2, 2, 2, 37, 41, 23, 23, for $i = 0, 1, 2, \ldots, 46$, where subscripts are considered modulo 47.

Solution to (b) by Kevin Ford, University of Illinois, Urbana, IL, and Sergei Konyagin, Moscow State University, Moscow, Russia. For integer N, let I be the integer interval $[-\alpha, \beta]$ of size N, where $\alpha = \beta = \frac{N-1}{2}$ when N is odd and $\beta = \alpha + 1 = \frac{N}{2}$ when N is even. For large enough N, we will specify a set $P = \{p_1, \cdots, p_j\}$ of primes (with $p_1 = 2$ and $p_2 = 3$), a residue $a_i \pmod{p_i}$ for each $p_i \in P$, and distinct numbers $x_2, \ldots, x_j, y_2, \ldots, y_j$ in I to satisfy the following properties:

(i) every $n \in I$ satisfies $n \equiv a_i \pmod{p_i}$ for some i,
(ii) $x_i \equiv y_i \equiv a_i \pmod{p_i}$ for each i,
(iii) each $x_i \equiv a_1 \pmod 2$, and
(iv) for each i, $y_i \equiv a_1 \pmod 2$ or $y_i \equiv a_2 \pmod 3$.

Given these properties, choose k such that $k \equiv -a_i \pmod{p_i}$ for all $p_i \in P$, and let $B = \{k + n : n \in I\}$. Note that $p_i \mid (k + n)$ if and only if $n \equiv a_i \pmod{p_i}$. By (i), every element of B is divisible by a prime in P. Reorder the indices $3, \ldots, r$ so that $y_i \equiv a_2 \pmod 3$ for $2 \le i \le r$ and $y_i \equiv a_1 \pmod 2$ for $r + 1 \le i \le j$ (note that this also holds for $i = 1$ by (ii)). Here $3 \le r \le j$. Let $S = \{x_2, \ldots, x_j\} \cup \{y_2, \ldots, y_j\}$. Let $B' = \{k + n : k + n \in B, n \notin S\}$. Let E be a list of the even elements of B' in some order. For each i, set $u_i = k + x_i$ and $v_i = k + y_i$. By (ii) and (iii), $2 \mid u_i$ and $p_i \mid u_i$ for all i. In particular, $6 \mid u_2$. By (ii) and (iv), $3 \mid v_i$ and $p_i \mid v_i$ for $2 \le i \le r$ and $2p_i \mid v_i$ for $r + 1 \le i \le j$. For $2 \le i \le j$, let Q_i denote a list $u_i, b_1, \ldots, b_s, v_i$, where $b_1, \ldots, b_s$ are the elements of B' whose smallest prime factor is p_i (it may happen that $s = 0$ and this central sublist is empty). In particular, every element of the list Q_i is divisible by p_i. Let R_i be the list Q_i for $r + 1 \le i \le j$ and for $2 \le i \le r$ with $r - i$ odd, and let R_i be the reverse of Q_i for $2 \le i \le r$ with $r - i$ even. The concatenation of the lists $E, R_2, \ldots, R_j$ now has the desired cyclic property, since 2 or 3 is a common factor at the boundaries of the sublists.

It remains to construct the desired sets. The set P will contain all primes that are at most $N/4$ together with some larger primes. Let $a_1 = 1$. For $3 \le p_i \le N/4$, let $a_i = 0$, $x_i = -p_i$, and $y_i = p_i$. Every $n \in I$ lies in at least one residue class of the form $a_i \pmod{p_i}$ except for $n \in M = I \cap \{\pm 2^d : d \ge 1\}$. Write $M = \{m_{t+1}, \ldots, m_j\}$, where t is the number of primes $\leq N/4$, and $|m_i| \geq |m_h|$ for $i < h$. We will find distinct primes $p_{t+1}, \dots, p_j$, each $> N/4$, and put $a_i = m_i$. Then (i) is satisfied.

There is flexibility in choosing p_i, x_i, y_i, but for large N we use the following. If $|m_i| \leq \frac{N}{16}$, we take

$$(1) \qquad x_i = m_i - p_i,\ y_i = m_i + p_i, \quad \tfrac{5N}{16} < p_i < \tfrac{7N}{16}.$$

If $|m_i| > \frac{N}{16}$, we take

$$(2) \quad \max\left(\tfrac{N}{4}, \tfrac{N}{8} + \tfrac{|m_i|}{2}\right) < p_i < \tfrac{N}{4} + \tfrac{|m_i|}{2}, \quad (x_i, y_i) = \begin{cases} (m_i - p_i, m_i - 2p_i) & m_i > 0 \\ (m_i + p_i, m_i + 2p_i) & m_i < 0 \end{cases},$$

and also impose the conditions

$$(3) \qquad p_i \equiv 11|m_i| \pmod{15}, \quad p_i - |m_i| \neq \pm 5.$$

By (1) and (2), each x_i, y_i lies in I and (ii) and (iii) hold. By (1) and (3), $y_i \equiv 1 \pmod 2$ when $|m_i| \leq \frac{N}{16}$ and $y_i \equiv 0 \pmod 3$ when $|m_i| > \frac{N}{16}$. Thus (iv) follows. By (1) and (2), $|y_i| > N/4$ for $i \geq t+1$ and $|x_i| > N/4$ when $|m_i| \leq \frac{N}{16}$. If $|m_i| > \frac{N}{16}$, then $|x_i| \leq \frac{N}{4}$, but (3) implies that $5|x_i$ and $x_i \neq \pm 5$. Hence no member of $x_{t+1}, y_{t+1}, \dots, x_j, y_j$ equals any member of $x_2, y_2, \dots, x_t, y_t$. Therefore, if

$$(4) \qquad p_i \nmid (x_l - m_i),\ p_i \nmid (y_l - m_i) \qquad (t+1 \leq l < i \leq j)$$

then all the x_i, y_i will be distinct. For a given i, there are at most $2(i-t-1)$ primes p_i failing (4). Thus, we find appropriate $p_{t+1}, \dots, p_j$ if for each i there are at least $3(i-t-1)+1$ primes p_i satisfying the appropriate conditions (1), (2) and (3). There are at most 6 numbers m_i with $|m_i| > \frac{N}{16}$ and $|M| \leq \frac{2\log(N/2)}{\log 2}$. Thus, we succeed if there are at least 18 primes in every open interval of length $\frac{N}{32}$ contained in $(\frac{N}{4}, \frac{N}{2})$ in each reduced residue class modulo 15 and if there are at least $\frac{6\log(N/2)}{\log 2}$ primes in the open interval $(\frac{5N}{16}, \frac{7N}{16})$. For large N, this occurs by the Prime Number Theorem for progressions modulo 15. Using explicit bounds for prime counts (O. Ramaré and R. Rumely, Math. Comp. **65** (1996), 397–425), these conditions hold for $N \geq 82000$.

Editorial comment. The GCHQ Problems Solving Group also asserted that 47 is the smallest N for which a solution exists. They gave another solution with $N = 49$ starting at the smaller integer 21176048208324. Ward obtained a solution with $N = 200$.

Ford and Konyagin noted that a computer search for triples (p_i, x_i, y_i) for $t+1 \leq i \leq j$ reveals that such configurations are possible for $517 \leq N < 82000$ and also for $330 \leq N \leq$

507. For example, the following works for $N = 330$.

m_i	p_i	x_i	y_i
-128	83	-45	121
128	103	25	-78
-64	113	49	162
64	89	-25	153
-32	97	-129	65
32	101	-69	133
-16	107	-123	91
16	109	-93	125
-8	127	-135	119
8	139	-131	147
-4	149	-153	145
4	131	-127	135
-2	157	-159	155
2	163	-161	165

Part (a) also solved by GCHQ Problem Solving Group (U. K.), University of Louisiana–Lafayette Math Club, and J. T. Ward.