3-9-11 #20
2-13-15 #11

Smooth Numbers

Let $\Psi(x,y) = \#\{n \le x : P^+(n) \le y\}$

Standard sieve (e.g. Theorem BH.2) gives $\Psi(x,y) \ll x \frac{\log y}{\log x}$.

This is very poor, especially when y is very small.

Theorem Ψ. Uniformly for $(\log x)^3 \le y \le x$

$$\Psi(x,y) = x e^{-u\log u + O(u \log\log(3u))}, \quad u = \frac{\log x}{\log y}.$$

Proof First, the upper bound. Let $\alpha = 1 - \frac{\log u}{\log y} \in [\frac{2}{3}, 1]$. For $n \le x$,

$$\log x = \log\frac{x}{n} + \log n \ll \left(\frac{x}{n}\right)^\alpha + \sum_{p^\nu \| n} \log p.$$

Therefore, writing $n = p^\nu m$,

$$(\log x)\Psi(x,y) \ll \underbrace{\sum_{P^+(n)\le y} \left(\frac{x}{n}\right)^\alpha}_{S_1} + \underbrace{\sum_{p\le y} \log p \sum_{\substack{m \le x/p \\ P^+(m)\le y}} 1}_{S_2} + \underbrace{\sum_{p,\nu\ge 2} \log p \sum_{\substack{m \le x/p^\nu \\ P^+(m)\le y}} 1}_{S_3}.$$

We have

$$S_2 \le \sum_{\substack{m \le x \\ P^+(m)\le y}} \sum_{\substack{p \le y \\ p \le x/m}} \log p \ll \sum_{P^+(m)\le y} \min\left(y, \frac{x}{m}\right)$$
$$\ll \sum_{P^+(m)\le y} y^{1-\alpha}\left(\frac{x}{m}\right)^\alpha = u S_1,$$

and

$$S_3 \le \sum_{p,\nu\ge 2} \log p \cdot \sum_{P^+(m)\le y} \left(\frac{x/p^\nu}{m}\right)^\alpha = S_1 \sum_{p,\nu\ge 2} \frac{\log p}{p^{\nu\alpha}} \ll S_1.$$

Therefore,

$$(\log x)\Psi(x,y) \ll u S_1 \le u x^\alpha \prod_{p\le y} \left(1 - \frac{1}{p^\alpha}\right)^{-1} \ll u x^\alpha \exp\Big\{\sum_{p\le y} \frac{1}{p^\alpha}\Big\}.$$

For $0 \le z \le 1$ and $c \ge 0$,

$$e^{cz} = 1 + z\sum_{k=1}^\infty \frac{c^k z^{k-1}}{k!} \le 1 + e^c z,$$

hence $$\frac{1}{p^\alpha} = \frac{1}{p} e^{(\alpha-1)\log p} = \frac{1}{p} e^{\log u \cdot \frac{\log p}{\log y}} \le \frac{1}{p}\left\{1 + e^{\log u} \frac{\log p}{\log y}\right\}$$

and thus $$\sum_{p\le y} \frac{1}{p^\alpha} \le \sum_{p\le y} \frac{1}{p} + \frac{u}{\log y} \sum_{p\le y} \frac{\log p}{p} \le \log\log y + O(u).$$

We conclude that

$$(\log x)\Psi(x,y) \ll e^{O(u)}(\log y) x^\alpha \ll (x\log x) e^{O(u) - u\log u}.$$

Next, the lower bound. Let $u_0 = \lfloor e^4+1 \rfloor = 55$, and let $k = \lfloor u \rfloor$. We will show that

$$(\Psi_1) \qquad \Psi(x,y) \geq x e^{-u\log u - C_0 u \log\log(3u)} \qquad (x \geq x_0,\ \log^3 x \leq y \leq x)$$

for some large constants C_0, x_0. First, (Ψ_1) holds for $0 < u \leq u_0$:

Put $h = \lfloor u \rfloor + 1$, $z = x^{1/h}$, $I = (z^{1-\frac{1}{2h}}, z]$, and consider numbers with exactly h prime factors in I, the product of these is in $(x^{1-\frac{1}{2h}}, x]$. So if $n = p_1 \cdots p_h m \leq x$ with $p_1 < \cdots < p_h$, each $p_i \in I$, then $P^+(m) < z^{1/2}$. Therefore,

$$\Psi(x,y) \geq \sum_{\substack{p_1 < \cdots < p_h \\ p_i \in I\ \forall i}} \left\lfloor \frac{x}{p_1 \cdots p_h} \right\rfloor \geq \frac{x}{2}\frac{1}{h!} \sum_{p_1 \in I} \frac{1}{p_1} \sum_{\substack{p_2 \in I \\ p_2 \neq p_1}} \frac{1}{p_2} \cdots \sum_{\substack{p_h \in I \\ p_h \notin \{p_1, \ldots, p_{h-1}\}}} \frac{1}{p_h}$$

$$\geq \frac{x}{2h!}\left(\sum_{p \in I} \frac{1}{p} - \frac{h}{z^{1-1/2h}}\right)^h \geq \delta_h x, \text{ some } \delta_h > 0,\ x > x_0(h).$$

This shows (Ψ_1) for $k \leq u_0 - 1$, if C_0 large enough. Now proceed by induction on k. Say $k \geq u_0$. Put $\eta = \frac{1}{\log u}$, $I = (y^{1-\eta}, y]$ and consider integers $n = ab \leq x$, where b is the product of exactly k primes from I and $P^+(a) \leq y^{1-\eta}$. Evidently each product ab is unique. Also, for each b,

$$\frac{x}{b} \leq \frac{x}{(y^{1-\eta})^k} \leq \frac{y^u}{(y^{1-\eta})^{u-1}} = (y^{1-\eta})^w, \quad w = 1 + \frac{\eta}{1-\eta} u.$$

As $w \leq 1 + \frac{4}{3}\frac{u}{\log u} \leq \frac{2u}{\log u} \leq \frac{u}{2} < u-1$, by the induction hypothesis,

$$\Psi(x,y) \geq \sum_b \Psi\left(\frac{x}{b}, y^{1-\eta}\right) \geq x \sum_b \frac{1}{b} e^{-w\log w - C_0 w \log\log(3w)}.$$

Now

$$w\log w + C_0 w \log\log(3w) \leq \frac{2u}{\log u}\cdot \log u + \frac{C_0}{2} u \log\log(3u)$$
$$\leq \left(2 + \frac{C_0}{2}\right) u \log\log(3u).$$

Next,

$$\sum_b \frac{1}{b} = \frac{1}{k!} \sum_{\substack{p_1, \ldots, p_k \in I \\ \text{distinct}}} \frac{1}{p_1 \cdots p_k} \geq \frac{1}{k!}\left(\sum_{p \in I} \frac{1}{p} - \frac{k}{y^{1-\eta}}\right)^k$$

$$= \frac{1}{k!}\left(\log \frac{1}{1-\eta} + O\left(\frac{1}{\log^2 y}\right)\right)^k \quad \text{by Mertens, and } \frac{k}{y^{1-\eta}} < \frac{1}{\log^2 y} \left(\text{since } y \geq \log^3 x\right)$$

$$\geq \frac{1}{k!}\left(\frac{1}{2\log u}\right)^k \qquad (x \geq x_0)$$

$$\geq \left(\frac{e}{2k\log u}\right)^k \geq e^{-u\log u - C_1 u \log\log(3u)}, \quad C_1 = \text{some absolute constant}$$

Therefore,

$$\Psi(x,y) \geq x e^{-u\log u - (C_1 + 2 + C_0/2) u \log\log(3u)},$$

which proves (Ψ_1) if $C_0 > 2(C_1 + 2)$.

Theorem Ψ2

$$\Psi(x, \log x) \ll \exp\left(3\,\frac{\log x \log_3 x}{\log_2 x}\right).$$

<u>Proof</u> Each $n \leq x$ can be written as $n = m^r \prod_{p \leq \log x} p^{a(p)}$, $0 \leq a(p) \leq r-1$, where $r \in \mathbb{N}$ is an arbitrary parameter. The number of such n is

$$\leq x^{\frac{1}{r}} \cdot r^{\pi(\log x)} = \exp\left\{\frac{\log x}{r} + (\log r)(1+o(1))\frac{\log x}{\log_2 x}\right\}.$$

Choose $r = \lfloor \log_2 x \rfloor$.