Smooth Numbers

Let \( \Psi(x, y) = \# \{ n \leq x : P(n) \leq y \} \).

Standard sieve (e.g., Theorem B4.2) gives \( \Psi(x, y) \ll x \frac{\log y}{\log x} \).

This is very poor, especially when \( y \) is very small.

**Theorem.** Uniformly for \( (\log x)^3 \leq y \leq x \),

\[
\Psi(x, y) = x e^{-u \log y + O(u \log \log y)},
\]

where \( u = \frac{\log x}{\log y} \).

**Proof.** First, the upper bound. Let \( x = 1 - \frac{\log u}{\log y} \in \left[ \frac{2}{3}, 1 \right] \).

For \( n \leq x \),

\[
\log n = \log \frac{x}{n} + \log n \ll \frac{1}{n} + \sum_{p \mid m} \log p.
\]

Therefore, using \( n = \prod p^m \),

\[
(\log x) \Psi(x, y) \ll \sum_{p \mid m} \left( \frac{x}{p^m} \right)^x + \sum_{p \leq y} \log p \sum_{m \leq x/p^x} \frac{1}{m}.
\]

We have

\[
S_2 \ll \sum_{p \mid m \leq x} \frac{\log p}{p^m} \ll \sum_{p \mid m \leq y} \frac{\min(y, \frac{x}{m})}{p^m} \ll \frac{1}{y} \sum_{p \mid m \leq y} \frac{1}{y} = u S_1,
\]

and

\[
S_3 \ll \sum_{p \leq y} \log p \cdot \sum_{p \mid m \leq x} \left( \frac{x}{p^m} \right)^x = S_1 \sum_{p \mid m \leq y} \frac{\log p}{p^m} \ll S_1.
\]

Thus,

\[
(\log x) \Psi(x, y) \ll u S_1 \ll ux^x \prod_{p \leq y} \left( 1 - \frac{1}{p^x} \right)^{-1} \ll u x^x \exp \left\{ \sum_{p \leq y} \frac{1}{p^x} \right\}.
\]

For \( 0 \leq z \leq 1 \) and \( c \geq 0 \),

\[
e^cz = 1 + \sum_{k=1}^{\infty} \frac{c^k z^{k-1}}{k!} \leq 1 + e^c z,
\]

hence

\[
\frac{1}{p^x} = \frac{1}{p} e^{(x-1) \log p} = \frac{1}{p} e^{\log p \log y} \leq \frac{1}{p} \left\{ 1 + e^{\log p} \right\} \frac{\log p}{\log y}.
\]

Hence

\[
\frac{1}{p^x} \leq \sum_{p} \frac{u}{p^x} \sum_{p} \frac{\log p}{p^x} \leq \log \log y + O(u).
\]

We conclude that

\[
(\log x) \Psi(x, y) \ll e^{O(u)(\log y)} x^x \ll (x \log x) e^{O(u) - u \log u}.
\]
Next, the lower bound. Let \( U = \log u + 1 = 55 \), and let \( k = \log u \). We will show that
\[
\psi(x, y) = x e^{-\log u - C_0 u \log \log (3u)} \quad (x > x_0, \log x = y = x)
\]
for some large constants \( C_0, x_0 \). First, \( \psi \) holds for \( 0 < u < u_0 \).

Put \( h = \log u + 1, z = x^{1/h}, I = \left[ z^{1-\frac{1}{h}}, z \right] \), and consider numbers with exactly \( h \) prime factors in \( I \), the product of these is in \( \left( x^{1-\frac{1}{h}}, x \right] \). So if \( n = p_1 \ldots p_h \leq x \) with \( p_1 < \ldots < p_h \), each \( p_i \leq I \), then \( p_i^{1/h} < \sqrt[4]{2} \). Therefore,
\[
\psi(x, y) \geq \sum_{P \subseteq \cdots \subseteq P_h} \left( \frac{x}{P_h} \right) \geq \frac{x}{2h!} \left( \sum_{P \subseteq \cdots \subseteq P_h} \frac{1}{P} \right) \geq \frac{x}{2h!} \left( \sum_{P \subseteq \cdots \subseteq P_h} \frac{1}{P} \right)
\]
This shows \( \psi \) for \( k = u_0 - 1 \), if \( C_0 \) large enough. Now proceed by induction on \( k \). Say \( k = u_0 \).

Put \( \eta = \frac{1}{\log u} \), \( I = \left( y^{1-\eta}, y \right] \) and consider integers \( n = ab \leq x \), where \( b \) is the product of exactly \( k \) primes from \( I \) and \( p_i^{\eta} \leq y \). Evidently each product \( ab \) is unique. Also, for each \( b \),
\[
\frac{x}{b} \leq \frac{x}{(y^{1-\eta})^k} \leq \frac{y^u}{(y^{1-\eta})^{k-1}} = (y^{1-\eta})^w, \quad w = 1 + \frac{k}{2} u.
\]
As \( w < 1 + \frac{k}{2} u \leq \frac{2u}{\log u} \leq \frac{u}{2} < u - 1 \), by the induction hypothesis,
\[
\psi(x, y) \geq \sum_b \psi \left( \frac{x}{b}, y^{1-\eta} \right) \geq \sum_b \frac{1}{b} e^{-\log w - C_0 w \log \log (3w)}
\]
Now
\[
\log w + C_0 w \log \log (3w) \leq \frac{2u}{\log u} \log u + \frac{C_0}{z} \log (3u) \leq (2 + \frac{C_0}{z}) u \log \log (3u).
\]
Next, \( \frac{1}{b} = \frac{1}{k!} \sum_{P \subseteq \cdots \subseteq P_h} \frac{1}{P} \geq \frac{1}{k!} \left( \sum_{P \subseteq \cdots \subseteq P_h} \frac{1}{P} \right) \geq \frac{1}{k!} \left( \frac{1}{\log y} + O \left( \frac{1}{\log^2 y} \right) \right)^k \)
by Mertens, and \( \frac{k}{y^{1-\eta}} < \frac{1}{\log y}(y^{1-\eta}) \) (since \( y^{1-\eta} < \frac{1}{\log y}(y^{1-\eta}) \))
\[
\geq \frac{1}{k!} \left( \frac{1}{\log u} \right)^k \left( x > x_0 \right)
\]
This proves \( \psi \) if \( C_0 > 2(c_1^2) \).

Therefore,
\[
\psi(x, y) \geq x e^{-\log u - (c_1 + 2 + c_0/2) \log \log (3u)}
\]
which proves \( \psi \) if \( C_0 > 2(c_1 + 2) \).
Theorem 1.2

\[ \Psi(x, \log x) \sim \exp \left( 3 \frac{\log x \log_{2} x}{\log_{2} 2} \right). \]

Proof. Each \( n \leq x \) can be written as \( n = m^r \prod_{p \leq \log x} p^{a(p)}, 0 \leq a(p) \leq r - 1, \)

where \( r \in \mathbb{N} \) is an arbitrary parameter. The number of such \( n \) is

\[ \leq x^r \cdot r \pi(\log x) = \exp \left\{ \frac{\log x}{r} + (\log r)((\log 1)/2) \frac{\log x}{\log_{2} x} \right\}. \]

Choose \( r = \lfloor \log_{2} x \rfloor \).