Math 532 Homework Problems, Spring 2011

1. (10 points each) Use theorems from class to prove the following

(a) Suppose $1 \leq a \leq b$ and $(a, b) = 1$. Show that the number, $N$, of primes $p \leq x$ for which $ap + b$ is also prime satisfies

$$N \ll \frac{ab}{\phi(ab)} \frac{x}{\log^2 x}$$

uniformly in $a, b, x$, i.e., the constant implied by the $\ll$ symbol is independent of $a, b$ and $x$.

(b) Let $\Xi(x, y, z)$ denote the number of integers $n \leq x$ which have no prime factor in $(y, z]$. Prove that uniformly in $2 \leq y \leq z \leq x$, we have

$$\Xi(x, y, z) \ll \frac{x \log y}{\log z}.$$

2. (20 points) Suppose $F(x) \in \mathbb{Z}[x]$, $F = F_1 \cdots F_m$, where each $F_i$ has positive leading coefficient and is irreducible over $\mathbb{Q}$, no $F_i$ is a rational multiple of any other $F_j$, and for every prime $p$, there is an $n$ so that $p \nmid F(n)$. Prove that for some constant $C$, depending on $\deg(F)$, and for $x > x_0(F)$, $x_0(F)$ some constant depending on $F$, we have

$$|\{n \leq x : \forall i, \omega(F_i(n)) \leq C\}| \gg x \log^m x.$$

3. (10 points) **Almost primes in short intervals.** Prove that for every $\delta > 0$, there is a natural number $k$ so that whenever $x \geq x_0(\delta)$, then the interval $(x, x + x^\delta]$ contains an integer $q$ with $\Omega(q) \leq k$.

4. (15 points) As in the proof of Theorem $H_k$, let

$$H_1(x) = \sum_{n \leq x} \frac{\mu^2(n)}{\phi(n)}$$

Show that $H_1(x) = \log x + c + o(1)$, where

$$c = \gamma + \sum_p \frac{\log p}{p(p - 1)} = 1.332 \ldots$$

and $\gamma = 0.5772 \ldots$ is Euler’s constant.
5. (20 points) Using “elementary” methods, show that
\[ \exp\{c_1 \frac{\log x}{\log \log x}\} \leq \Psi(x, \log x) \leq \exp\{c_2 \frac{\log x}{\log \log x}\} \]
for some constants \(0 < c_1 < c_2\).

6. (10 points) Show that for every \(\alpha < \frac{1}{2}\), a positive proportion of primes \(p\) satisfy both
\(P^+(p-1) > p^\alpha\) and \(P^+(p+1) > p^\alpha\).

7. Let \(a_i > 0, b_i \in \mathbb{Z}\) for each \(i\), \(Q(n) = \prod_{i=1}^{k} (a_i n + b_i)\) has no fixed prime factor and
\((a_i, b_i) \neq (a_j, b_j)\) for \(i < j\). Let \(\nu_Q(p)\) be the number of solutions of \(Q(n) \equiv 0 \pmod{p}\),
\(g(p) = \nu_Q(p)/p\), \(h(d) = \prod_{p|d} \frac{g(p)}{1-g(p)}\), and \(\mathcal{S}(Q) = \prod_{p} (1 - g(p)) (1 - 1/p)^{-k}\). Define \(\rho_d\) by
\((\rho')\) for \(d \leq z\) (with \(\mathcal{S}(h)\) replaced by \(\mathcal{S}(Q)\)) and \(L_z(n; Q, l) = \sum_{d|Q(n)} \rho_d\).
(a) (10 points. GPY 4 analog) Find an asymptotic formula for
\[ \sum_{N < n \leq 2N} L_z(n; Q, l)^2 \]
uniformly for real \(l \in [0, k]\), \(2 \leq z \leq N^{1/2}(\log N)^{-2k}\).
(b) (20 points. GPY5 analog) Find an asymptotic formula for
\[ \sum_{N < n \leq 2N} 1_p(an + b)L_z(n; Q, l)^2 \]
in two cases, (i) if \((a, b) = (a_j, b_j)\) for some \(j\), \(BV(\theta)\) holds and \(0 < \varepsilon < \theta\). Your formula
should be uniform in \(0 \leq l \leq k\) and \(3 \leq z \leq N^{1/2}(\theta - \varepsilon)\); (ii) \((a, b) \neq (a_j, b_j)\) for all \(j\), and
\(Q'(n) = (an + b)Q(n)\) has no fixed prime factors (i.e., \(\nu_Q(p) < p\) for all primes \(p\)). Your
formula should be uniform for \(1 \leq l \leq k\), \(0 \leq b \leq \log^2 z\), and \(3 \leq z \leq N^{1/2}(\theta - \varepsilon)\).

Remarks. Please mention any calculations which are identical to those in the proof of
Theorems GPY4 and GPY5 (these do not need to be repeated).

(c) (15 points) Let \((a, q) = 1\) and \(q > 1\). Let \(t_1, t_2, \ldots\) be the sequence of primes \(\equiv a\)
(mod \(q\)). Show that if \(BV(\theta)\) holds for some \(\theta > \frac{1}{2}\), then there is a constant \(C = C(\theta, q)\)
so that \(t_{n+1} - t_n \leq C\) for infinitely many \(n\).
(d) (20 points) Show unconditionally that
\[ \liminf_{n \to \infty} \frac{t_{n+1} - t_n}{\log t_n} = 0. \]

8. (10 points) Using the linear sieve, show that \(\Omega(n^2 + 1) \leq 4\) for infinitely many \(n\).
9. (Towards Chen’s theorem). Let \( x \) be large, \( y = x^{1/3} \) and \( z = x^{1/8} \). Let
\[
w(n) = 1 - \frac{1}{2} \sum_{z < q \leq y} 1 - \frac{1}{2} \sum_{n = \prod p \leq y \atop z < p_1 < p_2 < p_3} 1.
\]
Herre, \( q, p_1, p_2, p_3 \) are primes.

(a) (5 points) Show that if \( P^{-}(n) > z \), \( n \) is squarefree and \( w(n) > 0 \), then \( \Omega(n) \leq 2 \).

(b) (15 points) Let \( A = \{ p + 2 : p \leq x \} \) and \( \mathcal{P} \) be the set of all odd primes. Show that
\[
\# \{ p \leq x : \Omega(p + 2) \leq 2 \} \geq \sum_{p \leq x \atop \Omega(p + 2) > z} w(p + 2) + O(x^{7/8})
\geq S(A, \mathcal{P}, z) - \frac{1}{2} \sum_{z < q \leq y} S(A_q, \mathcal{P}, z) - \frac{1}{2} S(B, \mathcal{P}, y) + O(x^{7/8}),
\]
where \( B = \{ p_1 p_2 p_3 - 2 \leq x : z < p_1 \leq y < p_2 < p_3 \} \).

(c) (5 points) Show that
\[
S(A, \mathcal{P}, z) \geq \left( \frac{e^\gamma \log 3}{2} + o(1) \right) \frac{xV(z)}{\log x},
\]
where \( V(z) = \prod_{2 < p \leq z} \frac{p-2}{p-1} \).

(d) (20 points) Show that
\[
\sum_{z < q \leq y} S(A_q, \mathcal{P}, z) \leq \left( \frac{e^\gamma \log 6}{2} + o(1) \right) \frac{xV(z)}{\log x}.
\]