Problem 1
Let $B$ be the constant from the Hadamard product for $\xi(s)$.

(a) Show that $B = - \sum_{\substack{\zeta(\beta + i\tau) = 0 \\ 0 < \beta < 1 \\ \tau > 0}} \frac{2\beta}{\beta^2 + \tau^2}$.

(b) Use (a) to show that $\zeta(\sigma + i\tau) \neq 0$ in the region $0 < \sigma < 1$ and $|\tau| \leq 6$.

Solution
(a) Recall from class the formula
$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{1 - s - \rho} \right),$$
the sum over all nontrivial zeros of $\zeta(s)$. By the functional equation, we also have
$$\frac{\xi'(s)}{\xi(s)} = - \frac{\xi'(1 - s)}{\xi(1 - s)} = -B - \sum_{\rho} \left( \frac{1}{1 - s - \rho} + \frac{1}{\rho} \right) = -B - \sum_{\rho} \left( \frac{1}{1 - s - \rho} + \frac{1}{\rho} \right),$$
the second inequality following from the fact that $\zeta(\rho) = 0$ if and only if $\zeta(\bar{\rho}) = 0$. Equating these two expressions for $\xi'(s)/\xi(s)$ we get
$$2B = - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{1 - s - \rho} + \frac{1}{\rho} \right).$$
As this is true for any $s$, setting $s = 1$ gives
$$2B = - \sum_{\rho} \left( \frac{1}{1 - \rho} + \frac{1}{\rho} \right).$$
Now $1 - \rho$ is a zero if and only if $\rho$ is, so we can write this as
$$B = - \lim_{T \to \infty} \sum_{|\Im \rho| \leq T} \frac{1}{\rho}.$$
(b) Recall from class that
\[ B = -\gamma/2 - 1 + (1/2) \log(4\pi) = -0.0230957 \ldots, \]
where here \( \gamma = 0.5772\ldots \) is Euler’s constant. Now if \( \rho = \beta + i\tau \) is a nontrivial zero of \( \zeta(s) \), then by part (a),
\[ \frac{2\beta}{\beta^2 + \tau^2} \leq -B \leq 0.0231. \]
Furthermore, \( 1 - \beta + i\tau \) is also a zero, thus we may assume that the zero has \( 1/2 \leq \beta < 1 \). If \( |\tau| \geq 6 \) then
\[ \frac{2\beta}{\beta^2 + \tau^2} > \frac{1}{1 + 6^2} = \frac{1}{37} > 0.027, \]
a contradiction.

**Problem 2**

(a) Show that \( \zeta(-1) = -1/12 \);

(b) Show that for every \( k \in \mathbb{N} \), \( \zeta(s) > 0 \) for \( -4k < s < -4k + 2 \) and \( \zeta(s) < 0 \) for \( -4k + 2 < s < -4k + 4 \).

(c) Show that in the region \( |s - 1| \leq 10 \) we have \( \zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|) \).

**Solution**

(a) Plug in \( s = 1 \) into the functional equation for \( \zeta(s) \):
\[ \pi^{1/2} \Gamma(-1/2)\zeta(-1) = \pi^{-1} \Gamma(1)\zeta(2). \]
Now \( \Gamma(1) = 1 \) and \( (-1/2)\Gamma(-1/2) = \Gamma(1/2) = \sqrt{\pi} \), thus
\[ \zeta(-1) = \frac{1}{2\pi^2} \zeta(2) = -\frac{1}{12}. \]

(b) We have \( \zeta(s) \) is real for real \( s < 1 \). We know that \( \zeta(s) < 0 \) for \( s < 1 \) and close to \( 1 \) due to the pole at \( s = 1 \) (see part (c)). The result now follows since \( \zeta(s) \) has simple zeros at \( s = -2, -4, -6, \ldots \) and thus changes sign precisely at these points.

(c) It suffices to show that the constant term in the Laurent expansion of \( \zeta(s) \) at \( s = 1 \) is equal to \( \gamma \). From class, we have for \( \sigma = \Re s > 0 \),
\[ \zeta(s) = \frac{1}{s-1} + 1 - s \int_1^\infty \{t\} t^{-s-1} dt. \]
Thus,
\[ \lim_{s \to 1} \zeta(s) = \frac{1}{s-1} = 1 - \int_1^\infty \{t\} t^{-2} dt, \]
which equals \( \gamma \) by the proof of the asymptotic for \( \sum_{n \leq x} 1/n \).

**Problem 3**

Let \( \theta \) be the supremum of the real parts of zeros of \( \zeta(s) \). Show that if \( \theta < 1 \) then \( \psi(x) = x + O(x^\theta \log^2 x) \) and \( \pi(x) = \text{li}(x) + O(x^\theta \log x) \).
Solution

Recall the explicit formula for $\psi(x)$ proved in class:

$$\psi(x) = x - \sum_{|\Im \rho| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right).$$

Take $T = x$, note that $|x^\rho| = x^{\Re \rho} \leq x^\theta$ by hypothesis. Finally, by a result from class, the number of zeros with imaginary part in $[n, n+1]$ is $O(\log n)$ and it follows that

$$\sum_{|\Im \rho| \leq T} \frac{1}{|\rho|} \ll \sum_{n \leq x} \frac{O(\log n)}{n} \ll \log^2 x.$$

This completes the proof of the first part. The second part follows from a general result proved in class about error terms in the various forms of the PNT.

Problem 4

(10 points) Prove that if $\zeta(\sigma + i\tau) \neq 0$ for $\sigma \geq 1 - A/\log |\tau|$ and $|\tau| \geq 6$, then

$$\left|\frac{1}{\zeta(\sigma + i\tau)}\right| \ll A \log(|\tau| + 2) \quad \left(\sigma \geq 1 - \frac{A}{\log(|\tau| + 2)}, \text{ all } \tau\right).$$

Hint: break into two cases. For $\sigma \geq 1 + (A/2)/L$, $L = \log(|\tau| + 2)$, use the bound $|1/\zeta(s)| \leq \zeta(\sigma)$. In the other case $1 - A/(2L) \leq \sigma \leq 1 + A/(2L)$, use bounds on $\zeta'(s)/\zeta(s)$ together with

$$\log \zeta(A) - \log \zeta(B) = \int_{B}^{A} \frac{\zeta'(s)}{\zeta(s)} ds.$$

Solution

NOT GRADED. For $\sigma \geq 1 + (A/2)/L$, $L = \log(|\tau| + 2)$, we have

$$|1/\zeta(s)| \leq \sum_{n=1}^{\infty} \left|\frac{\mu(n)}{n^s}\right| \leq \zeta(\sigma) = \frac{1}{\sigma - 1} + O(1) = O(L)$$

by Theorem 5.1. Now suppose that $s = \sigma + i\tau$ with $1 - A/(2L) \leq \sigma \leq 1 + A/(2L)$, and let $C = 1 + A/(2L) + i\tau$. On the line between $s$ and $C$, we have the bound $|\zeta'(s)/\zeta(s)| \ll L$. This is classical, but doesn’t follow quickly from results proved in class, as I originally thought. From this bound we get

$$|\log \zeta(s) - \log \zeta(C)| = \left|\int_{C}^{s} \frac{\zeta'(z)}{\zeta(z)} dz\right| \ll L|C - s| \ll 1.$$

Therefore,

$$\frac{1}{\zeta(s)} = \frac{1}{\zeta(C)} e^{\log \zeta(C) - \log \zeta(s)} \ll \frac{1}{\zeta(C)} \ll L.$$

Problem 5

(20 points) Let $M(x) = \sum_{n \leq x} \mu(n)$. Using Perron inversion with $1/\zeta(s)$, show that

$$M(x) = O(x e^{-c\sqrt{\log x}}),$$

where $c$ is some positive constant. You may use the conclusion of the previous problem.
Solution

Let \( x \geq 10, x - 1/2 \in \mathbb{Z} \), and suppose \( 1 \leq T \leq x \). First, apply the truncated version of Perron’s formula, with \( f(n) = \mu(n) \), \( F(s) = 1/\zeta(s) \) and \( c = 1 + 1/\log x \). Since \( |\mu(n)| \leq 1 \) for all \( n \), we obtain

\[
|M(x) - I| \ll \frac{x}{T} \sum_{n=1}^{\infty} \frac{1}{n^c \log(x/n)} + 1,
\]

where

\[
I = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{\zeta(s)} \frac{ds}{s}.
\]

When \( n > (3/2)x \) or \( n \leq x/2, |\log(x/n)| \gg 1 \), so the contribution to the above sum from such \( n \) is at \( \ll \zeta(c) \ll \log x \). As in the proof of the explicit formula for \( \psi(x) \), \( |\log(x/n)| \geq n_0/n \) if \( x < n < (3/2)x \) and \( |\log(x/n)| \geq n_0/n^2 \) for \( x/2 < n < x \). Thus, these terms in the above sum contribute at most \( O(\log x) \). Therefore

\[
|M(x) - I| \ll 1 + \frac{x \log x}{T}.
\]

To bound \( I \), we replace the vertical segment \([c - iT, c + iT] \) with three segments: \( D_1 \) is the segment from \( c - iT \) to \( b - iT \), \( D_2 \) is the segment from \( b - iT \) to \( b + iT \), and \( D_3 \) is the segment from \( b + iT \) to \( c + iT \). Here \( b = 1 - \frac{K}{T}, L = \log(T + 2) \), where \( K \) is chosen smaller than \( B \) and so small that there are no zeros of \( \zeta \) in the region \(|3s| \leq T, \Re s \geq b \). Since there are no poles of \( x^s/(s\zeta(s)) \) inside the rectangle formed by the four segments, \( I \) equals the integral over \( D_1 \cup D_2 \cup D_3 \). On all 3 segments, \( |1/\zeta| \ll L \) by the previous problem. Since \( c - b \ll 1 \), the integral over \( D_1 \cup D_3 \) is \( O(xL/T) \) and the integral over \( D_2 \) is

\[
\ll Lx^b \int_{-T}^{T} \frac{dt}{|t|+1} \ll L^2x^b.
\]

Therefore,

\[
|I| \ll \frac{xL}{T} + L^2x^b.
\]

From (1) we conclude that

\[
|M(x)| \ll (\log x) \left(1 + \frac{x}{T} + x^b \right).
\]

Lastly, we take the optimal choice \( T = \exp\{\sqrt{K \log x} \} \) to complete the proof.

Problem 6

The Vinogradov-Korobov zero-free regions for \( \zeta(s) \). Assume that for some constant \( B > 0 \),

\[
|\zeta(\sigma + i\tau)| \ll \begin{cases} T^{B(1-\sigma)^{3/2}} (\log T)^{2/3} & \left( \frac{1}{2} \leq \sigma \leq 1 \right) \\ (\log T)^{2/3} & (\sigma > 1), \end{cases}
\]

where \( T = |t| + 2 \). Prove that there is a constant \( c_0 > 0 \) so that we have the zero-free region

\[
\zeta(\sigma + i\tau) \neq 0 \quad \left( |\tau| \geq 6, \quad \sigma \geq 1 - \frac{c_0}{(\log |\tau|)^{2/3}(\log \log |\tau|)^{1/3}} \right).
\]

4
Solution

Suppose $\zeta(\beta + i\tau) = 0$ with $\tau > 6$. Let $L = \log(2\tau + 10)$. We claim that if $c > 0$ is sufficiently small then

$$1 - \beta \geq \frac{c}{L^{2/3}(\log L)^{1/3}}.$$  

This then implies (2) with some positive $c_0$ since $L \asymp \log \tau$ and $\log L \asymp \log \log \tau$. Suppose that $c$ is sufficiently small in terms of $B$ and (3) is false. Let

$$\sigma_0 = 1 + 10(1 - \beta), \quad s_0 = \sigma_0 + i\tau,$$
$$r = \sigma_0 - \beta = 11(1 - \beta),$$
$$R = \left(\frac{\log L}{L}\right)^{2/3}.$$  

If $c$ is small enough then $4r \leq R \leq 1$. By the proof of the classical zero-free region proved in class (using a 3-4-1 inequality and Heath-Brown’s zero detector), we have

$$0 \leq \frac{3}{\sigma_0 - 1} - 4\left(\frac{1}{r} - \frac{r}{R^2}\right) - \frac{1}{\pi R} \int_{-\pi/2}^{\pi/2} (\cos \theta) \left(4 \log |\zeta(s_1(\theta))| + \log |\zeta(s_2(\theta))|\right) d\theta,$$

where

$$s_1(\theta) = \sigma_0 + i\tau + Re^{i\theta}, \quad s_2(\theta) = \sigma_0 + 2i\tau + Re^{i\theta}.$$  

As in the class proof of the classical zero-free region, the contribution of the integral in (4) from $|\theta| \leq \pi/2$ is at most

$$\frac{10}{\pi R} \log \left(1 + \frac{1}{\sigma_0 - 1}\right) \leq \frac{10}{\pi R} \left(\frac{2}{3} \log L + \frac{1}{3} \log \log L + O(1)\right).$$

By hypothesis, when $\pi/2 \leq \theta \leq 3\pi/2$ and $s \in \{s_1(\theta), s_2(\theta)\}$ we have

$$|\zeta(s)| \ll (2\tau + 10)^{BR^{3/2} L^{2/3}}.$$  

So the contribution of the part of the integral in (4) with $\pi/2 \leq \theta \leq 3\pi/2$ is at most

$$\frac{10}{\pi R} \left(BR^{3/2} L + (2/3) \log L + O(1)\right).$$

Thus, from (4) and recalling that $R \geq 4r$, we have

$$0 \leq \frac{3}{\sigma_0 - 1} - \frac{15/4}{r} + \frac{10}{\pi R} \left(BR^{3/2} L + (4/3) \log L + (1/3) \log \log L + O(1)\right).$$

Next,

$$\frac{3}{\sigma_0 - 1} - \frac{15/4}{r} = \frac{1}{1 - \beta} \left(\frac{3}{10} - \frac{15}{44}\right) = -\frac{9/220}{1 - \beta},$$

and thus

$$\frac{9/220}{1 - \beta} \leq \left(\frac{10B}{\pi} + \frac{40}{3\pi}\right) L^{2/3}(\log L)^{1/3} + O(L^{2/3}(\log L)^{-2/3}(\log \log L)^{1/3})$$

$$\ll B L^{2/3}(\log L)^{1/3}.$$  

We get a contradiction if $c$ is small enough in terms of $B$.  

5
Problem 7
Best-known error term in the PNT. Assume a zero-free region (2), where $c_0 > 0$ is constant. Prove that there is a constant $c_1 > 0$ such that

$$\psi(x) = x + O\left(x \exp \left\{ -c_1 (\log x)^{3/5} (\log \log x)^{-1/3} \right\}\right).$$

Solution
As in Problem 3, start with the explicit formula for $\psi(x)$ proved in class:

$$\psi(x) = x - \sum_{|\Im \rho| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right).$$

We will choose $T \in [10, x]$ later. As in problem 3,

$$\sum_{|\Im \rho| \leq T} \frac{1}{|\rho|} \ll \log^2 T \leq \log^2 x.$$ 

Also, by Problem 6, the real part of $\rho$ satisfies

$$\Re \rho \leq 1 - \xi, \quad \xi = \frac{c}{(\log T)^{2/3} (\log \log T)^{1/3}}.$$ 

Hence,

$$\psi(x) = x + O\left((x \log^2 x) \left(\frac{1}{T} + x^{-\xi}\right)\right).$$

We choose $T$ such that $T \approx x^\xi$, specifically

$$T = \exp \left\{ (c \log x)^{3/5} (\log \log x)^{-1/5} \right\}.$$ 

We than have, for some positive $c'$,

$$\psi(x) - x \ll x e^{-c'(\log x)^{3/5} (\log \log x)^{-1/5}} \log^2 x,$$

and by reducing $c'$ we can remove the $\log^2 x$ factor.