Problem 1
Consider the function \( f_i : [10, \infty) \to \mathbb{R} \) given by
\[
\begin{align*}
    f_1(x) &= x^{1/\log \log x}, \\
    f_2(x) &= e^{\sqrt{\log x}}, \\
    f_3(x) &= x, \\
    f_4(x) &= (\log x)^{100}, \\
    f_5(x) &= \sqrt{x}, \\
    f_6(x) &= e^x, \\
    f_7(x) &= x, \\
    f_8(x) &= (\log x)^{100}, \\
    f_9(x) &= \log \log x.
\end{align*}
\]

Place the functions in order, where \( f_i \) precedes \( f_j \) if \( f_i(x) = o(f_j(x)) \) as \( j \to \infty \). Provide complete proofs of your ordering.

Solution
The ordering is
\[
f_9 \preceq f_4 \preceq f_2 \preceq f_1 \preceq f_5 \preceq f_8 \preceq f_7 \preceq f_3 \preceq f_6.
\]
All of these are easy consequence of the following facts, themselves proved using L’Hospital’s rule:
\[
\begin{align*}
    x &= o(e^{ax}) \quad (a > 0 \text{ fixed}), \\
    \log x &= o(x^a) \quad (a > 0 \text{ fixed}), \\
    \log \log x &= o((\log x)^a) \quad (a > 0 \text{ fixed}).
\end{align*}
\]

Problem 2
Let \( f : [0, \infty) \to \mathbb{R} \) and \( g : [0, \infty) \to [0, \infty) \) be integrable functions, and set
\[
F(x) = \int_0^x f(y)dy, \quad G(x) = \int_0^x g(y)dy.
\]

(a) Show that if \( f(x) = o(g(x)) \) \( (x \to \infty) \) and if \( G(x) \to \infty \) as \( x \to \infty \), then then
\[
F(x) = o(G(x)) \quad (x \to \infty).
\]

(b) Find an example of functions \( f, g \) satisfying \( f(x) = o(g(x)) \) but \( F(x) \neq o(G(x)) \) as \( x \to \infty \).

Solution
(a) Let \( \epsilon > 0 \) be given. Since \( f(x) = o(g(x)) \) there exists a \( y_0(\epsilon) > 0 \) such that
\[
|f(y)| \leq \epsilon g(y) \quad (y \geq y_0(\epsilon)).
\]
Then, for $x \geq y_0(\epsilon)$,

$$|F(x)| = \left| \int_0^{y_0(\epsilon)} f(y) dy + \int_{y_0(\epsilon)}^x f(y) dy \right|$$

$$\leq \int_0^{y_0(\epsilon)} f(y) dy + \int_{y_0(\epsilon)}^x \epsilon g(y) dy$$

$$\leq \int_0^{y_0(\epsilon)} f(y) dy + \frac{\epsilon}{2} G(x).$$

Our additional assumption that $G(x) \to \infty$ as $x \to \infty$ implies that for sufficiently large $x$, the right side is $\leq \epsilon G(x)$, and this shows that $F(x) = o(G(x))$ as $x \to \infty$.

(b) An example is given by $f(y) = e^{-2y}$ and $g(y) = e^{-y}$. Since $f(y)/g(y) = e^{-y} \to 0$ as $y \to \infty$, we have $f(y) = o(g(y))$. On the other hand, $F(x) = (1 - e^{-2x})/2$ and $G(x) = 1 - e^{-x}$ converge to $1/2$ and $1$, respectively, as $x \to \infty$, so $F(x) = o(G(x))$ certainly does not hold.

**Problem 3**

Prove the cancellation property of arithmetic functions: if $f, g, h$ are arithmetic functions satisfying $f \ast g = f \ast h$, and $f$ is not identically zero, then $g = h$. Thus, combined with the other properties proved in class, the set $A$ of arithmetic functions together with the + and $\ast$ operators forms an integral domain.

**Solution**

We verify an equivalent condition: that whenever $f \ast g = 0$, the identically zero function, then $f = 0$ or $g = 0$. Assume that there are nonzero $f, g \in A$ satisfying $f \ast g = 0$. Let $a = \min\{n \in \mathbb{N} : f(n) \neq 0\}$ and $b = \min\{n \in \mathbb{N} : g(n) \neq 0\}$. The numbers $a, b$ exist by the fact that $f \neq 0$, $g \neq 0$. Consider

$$(f \ast g)(ab) = \sum_{d|ab, d > 0} f(d)g(n/d)$$

On the right side, all summands are zero with $d < a$ and with $n/d < b$ (that is, $d > n/b = a$), by the definition of $a$ and $b$. So only the term with $d = a$ survives and we have $(f \ast g)(ab) = f(a)g(b)$, which is nonzero. This is a contradiction.

Finally, if $f \ast g = f \ast h$, then $f \ast (g - h) = 0$. If $f$ is not identically zero, it follows that $g - h = 0$, that is, $g = h$.

**Problem 4**

Suppose $f : \mathbb{N} \to \mathbb{C}$ is multiplicative.

(a) Prove that $\sum_{d|n} f(d) = \prod_{p^n|n} (1 + f(p) + f(p^2) + \cdots + f(p^n))$.

(b) Suppose $\sum_{n=1}^{\infty} |f(n)|$ converges. Show that

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots);$$

that is, prove rigorously that the partial product $\prod_{p \leq x} (1 + f(p) + \cdots)$ converges to $\sum_{n=1}^{\infty} f(n)$. Hint: compare the partial product with $\sum_{n \leq x} f(n)$.
(c) Define \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) the Riemann zeta function. Show that if \( \Re s > 1 \) then
\[
\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.
\]

(d) Using (c), show that if \( \Re s > 1 \) then
\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.
\]

Solution

(a) Let \( n = p_1^{e_1} \cdots p_k^{e_k} \). The product on the right side is equal to
\[
\sum_{a_1=0}^{e_1} \cdots \sum_{a_k=0}^{e_k} f(p_1^{a_1}) \cdots f(p_k^{a_k}) = \sum_{a_1=0}^{e_1} \cdots \sum_{a_k=0}^{e_k} f(p_1^{a_1} \cdots p_k^{a_k}) = \sum f(d).
\]

(b) Since \( \sum f(n) \) converges absolutely, so does the infinite product on the right side. Let \( x \) be a large real number. By the multiplicativity of \( f \), we have
\[
\prod_{p \leq x} (1 + f(p) + f(p^2) + \cdots) = \sum_{n \in S(x)} f(n),
\]
where \( S(x) \) is the set of integers (including 1) composed only of primes which are \( \leq x \). Thus
\[
\prod_{p \leq x} (1 + f(p) + f(p^2) + \cdots) = \sum_{n \leq x} f(n) + \sum_{n > x, n \in S(x)} f(n)
\]
\[
= \sum_{n=1}^{\infty} f(n) - \sum_{n > x, n \notin S(x)} f(n)
\]
\[
= \sum_{n=1}^{\infty} f(n) + O \left( \sum_{n > x} |f(n)| \right)
\]
\[
= \sum_{n=1}^{\infty} f(n) + o(1)
\]
as \( x \to \infty \). The desired result follows immediately.

(c) Apply (b) to the function \( f(n) = 1/n^s \). If \( s = \sigma + it \), where \( \sigma, t \in \mathbb{R} \) and \( \sigma > 1 \) then \( |f(n)| = n^{-\sigma} \) and hence \( \sum |f(n)| \) converges. Lastly,
\[
1 + f(p) + f(p^2) + \cdots = \sum_{k=1}^{\infty} \frac{1}{p^{ks}} = \frac{1}{1 - p^{-s}}.
\]

(d) By (c),
\[
\frac{1}{\zeta(s)} = \prod_{p} (1 - p^{-s}).
\]

Now apply (b) with the function \( f(n) = \mu(n)/n^s \) to see that
\[
\prod_{p} (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.
\]
Problem 5
Suppose $f$ is an arithmetic function and
\[ \sum_{n \leq x} f(n) \sim Ax \quad (x \to \infty), \]
where $A$ is a nonzero complex number.

(a) Show that
\[ \sum_{n \leq x} \frac{f(n)}{n} \sim A \log x \quad (x \to \infty). \]

(b) Find an example of an $f$ which shows that the converse is false, that is, $\sum_{n \leq x} f(n)/n \sim A \log x$ but $\sum_{n \leq x} f(n) \neq Ax$.

Solution
(a) Let
\[ S(x) = \sum_{n \leq x} f(n) = Ax + E(x), \quad T(x) = \sum_{n \leq x} \frac{f(n)}{n}, \]
where $E(x) = o(x)$ as $x \to \infty$. By partial summation,
\[ T(x) = A \log x + \frac{E(x)}{x} - E(1) + \int_1^x \frac{E(t)}{t^2} \, dt. \]
Noe $E(x)/x = o(1)$ and $E(1) = O(1)$. Since $\int_1^x dt/t \to \infty$ as $x \to \infty$, we invoke Problem 2 (a) above and conclude that the final integral is $o(\log x)$ as $x \to \infty$.

(b) Let $f(n) = n$ when $n$ is a power of two, and $f(n) = 0$ otherwise. We have
\[ \sum_{n \leq x} \frac{f(n)}{n} = \sum_{2^k \leq x} 1 = \frac{x}{\log 2} + O(1) \sim \frac{1}{\log 2} \log x \quad (x \to \infty), \]
but the function of $x$ given by
\[ \frac{1}{x} \sum_{n \leq x} f(n), \]
has jumps of size 1 at every power of two, and hence cannot tend to any limit as $x \to \infty$.

Problem 6
(a) Let $s > 1$ and $x \geq 1$ is a real number. Show that
\[ \sum_{n > x} \frac{1}{n^s} = \frac{1}{(s - 1)x^{s-1}} + O(x^{-s}). \]

Remark: the constant implied by $O-$ is absolute, that is, independent of $s$ and $x$.

(b) Show that $\lim_{s \to 1^+} (s - 1)\zeta(s) = 1$. 

Solution

(a) Euler summation gives

$$\sum_{x < n \leq z} \frac{1}{n^s} = \int_{x}^{z} \frac{dt}{t^s} - \frac{\{x\}}{x^s} + \int_{x}^{z} \{t\} (-s t^{-s-1}) dt.$$

Since $s > 1$, we may let $z \to \infty$ on both sides, obtaining

$$\sum_{x < n} \frac{1}{n^s} = \int_{x}^{\infty} \frac{dt}{t^s} + \frac{\{x\}}{x^s} + \int_{x}^{\infty} \{t\} (-s t^{-s-1}) dt.$$

The first integral equals $\frac{1}{(s-1)x^s}$, we have $\{x\} \in [0, 1]$ and

$$\int_{x}^{\infty} \{t\} (st^{-s-1}) dt \leq s \int_{x}^{\infty} t^{-s-1} dt = x^{-s}.$$

(b) Apply (a) with $x = 1$, get

$$\zeta(s) = 1 + \frac{1}{s-1} + O(1) = \frac{1}{s-1} + O(1).$$

The desired limit follows.