Problem 1
Consider the axioms for a Commutative Ring with Identity, as given in class. Below are three mathematical objects which fail at least one of these axioms. For each one, identify the axiom that fails (and show that the axiom fails by example). You do not need to prove that the other axioms hold, nor do you need to find all the axioms that fail.

(a) \( 3\mathbb{Z} = \{3n : n \in \mathbb{Z}\} \), with ordinary addition and multiplication.
(b) \( \frac{1}{5}\mathbb{Z} = \{\frac{n}{5} : n \in \mathbb{Z}\} \), with ordinary addition and multiplication.
(c) \( V_3(\mathbb{Z}) \), the set of 3-dimensional vectors \( \vec{v} = \langle a, b, c \rangle \) with \( a, b, c \in \mathbb{Z} \), with ordinary addition, and with multiplication defined by the cross-product \( \vec{v} \times \vec{w} \).

The following problems are about the set \( \mathcal{A} \) of arithmetic functions, that is, functions \( f : \mathbb{N} \to \mathbb{R} \). Define two operators on such functions, “+” being ordinary addition (\( h = f + g \) is the function that satisfies \( h(n) = f(n) + g(n) \) for all \( n \in \mathbb{N} \)), and \( * \) being the other operation, where the function \( f * g \) satisfies
\[
(f * g)(n) = \sum_{d|n, d>0} f(d)g(n/d) \quad (n \in \mathbb{N}).
\]
In the next two problems, we verify that \( (\mathcal{A}, +, *) \) is a Commutative Ring with Identity.

Problem 2
Prove that \( (\mathcal{A}, +, *) \) satisfies Closure, the Associative Laws, Commutative Laws and the Distributive Law.

Problem 3
(a) Find the Additive Identity function \( 0 \), and verify the additive identity and additive inverse rules.
(b) Find the Multiplicative Identity function \( 1 \) and verify the multiplicative identity rule.

Problem 4
Suppose \( S = \{0, 1, x, y\} \) is a field containing exactly 4 elements, with 0 being the additive identity and 1 the multiplicative identity.
(a) Determine the value of \( xy \) (with complete, detailed proof, of course). Use this to fill out the rest of the multiplication table of \( S \) (explaining each entry - you may be brief about some of them).
(b) Determine the value of \( x + y \). Then fill out the rest of the addition table for \( S \), giving a complete proof/explanation for each entry.