Math 347 Honors Spring 2018
Homework # 7
Solutions

Problem 1
For each relation $R$ on the set $S$, determine if it is an equivalence relation (with proof). If it is not, be sure to specify which property fails, with an example.

(i) $S = \mathbb{R}$; $(x,y) \in R$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$.
(ii) $S = \mathbb{Z}$; $(x,y) \in R$ if and only if $-1 \leq x - y \leq 1$.
(iii) $S$ is the set of all functions $f: \mathbb{R} \to \mathbb{R}$, and $(g,h) \in R$ if and only if $g-h$ is bounded.

Solution
(i) The reflexive property holds because $x = 2^0 x$ for all $x \in \mathbb{R}$. The symmetric property holds because if $x = 2^n y$ with $n \in \mathbb{Z}$, then $y = 2^{-n}x$. The transitive property holds: if $x = 2^n y$ and $y = 2^m z$ with $n,m \in \mathbb{Z}$, then $x = 2^{n+m} z$. Therefore $R$ is an equivalence relation.

(ii) The transitive property fails. We have $(0,1) \in R$ and $(1,2) \in R$ but $(0,2) \notin R$. So $R$ is not an equivalence relation.

(iii) This is an equivalence relation. For all $f : \mathbb{R} \to \mathbb{R}$, $(f,f) \in R$ because $f-f = 0$ is clearly a bounded function. Also, if $(f,g) \in R$, that is, if $f-g$ is bounded, this means there is some number $M$ so that for all $x \in \mathbb{R}, |f(x)-g(x)| \leq M$. But then
\[ |g(x)-f(x)| = |f(x)-g(x)| \leq M \]
as well, for every $x \in \mathbb{R}$, so $g-f$ is bounded, hence $(g,f) \in \mathbb{R}$. Finally, if $(f,g) \in R$ and $(g,h) \in R$, then $f-g$ and $g-h$ are bounded. So there are numbers $M, N$ so that for all $x \in \mathbb{R}, |f(x)-g(x)| \leq M$ and $|g(x)-h(x)| \leq N$. By the triangle inequality,
\[ |f(x)-h(x)| = |(f(x)-g(x)) + (g(x)-h(x))| \leq |f(x)-g(x)| + |g(x)-h(x)| \leq M + N \]
for every $x \in \mathbb{R}$. Thus, $f-h$ is bounded, i.e. $(f,h) \in R$.

Problem 2
Problem 7.14

Solution
Let $f : \mathbb{R} \to \mathbb{R}$. First, for all $g \in S$, and for all $x$, $g(x) - g(x) = 0 \leq |f(x)|$, so $f \sim f$. If $g,h \in O(f)$, then there are positive real numbers $c,a$ so that for all $x > a$, $|g(x)-h(x)| \leq c |f(x)|$. But this implies $|h(x) - g(x)| \leq c |f(x)|$ for all $x > a$, so $h-g \in O(f)$. Finally, suppose $g-h \in O(f)$ and $h-k \in O(f)$. Then for some positive real numbers $a_1,a_2,c_1,c_2$, we have
\[ \forall x > a_1 : |g(x) - h(x)| \leq c_1 |f(x)|, \quad \forall x > a_2 : |h(x) - k(x)| \leq c_2 |f(x)|. \]
Let $a = \max(a_1,a_2)$ and $c = c_1 + c_2$. By the Triangle Inequality, for all $x > a$,
\[ |g(x)-k(x)| = |(g(x)-h(x)) + (h(x)-k(x))| \leq |g(x)-h(x)| + |h(x)-k(x)| \leq (c_1+c_2) |f(x)| = c |f(x)|. \]
Therefore, $g-k \in O(f)$. Since the relation satisfies the reflexive, symmetric and transitive properties, $R$ is an equivalence relation.
Problem 3

Problem 7.18. Solution to $2n^2 + n \equiv 0 \pmod{p}$ when $p$ is an odd prime.

Solution

We factor $2n^2 + n$ as $n(2n + 1)$ and ask when the product is a multiple of $p$. Since $p$ is prime, this occurs if and only if $p|n$ or $p|(2n + 1)$, i.e. either $n \equiv 0 \pmod{p}$ or $2n + 1 \equiv 0 \pmod{p}$. The second congruence is $2n \equiv p - 1 \pmod{p}$. Since $p$ is odd, $\gcd(2, p) = 1$ and therefore we can divide by 2, yielding $n \equiv (p - 1)/2 \pmod{p}$.

Problem 4

Problem 7.30.

Solution

(a) Suppose $n$ has a decimal expansion $n = d_k10^k + d_{k-1}10^{k-1} + \cdots + d_110 + d_0$, where $d_k, \ldots, d_0$ are the digits of $n$. Since $10 \equiv 1 \pmod{3}$, $10^m \equiv 1 \pmod{3}$ for all positive integers $m$. Thus,

$$n \equiv d_k + d_{k-1} + \cdots + d_1 + d_0 \pmod{3}.$$

The right side of the congruence is the sum of the digits of $n$, so clearly that sum is 0 mod 3 if and only if $n \equiv 0 \pmod{3}$.

(b) Suppose $x - 1$ and $x + 1$ are both prime. Since there is only one even prime, namely 2, $x$ must be even. That is, $2|x$. If $x$ is not divisible by 3, then $x \equiv 1 \pmod{3}$ or $x \equiv 2 \pmod{3}$. In the first case, $3|(x - 1)$, so $x - 1$ won’t be prime unless $x - 1 = 3$ (then $x - 1 = 5$ is prime). In the second case, $3|(x + 1)$, which isn’t prime unless $x + 1 = 3$ (but then $x - 1 = 1$, which isn’t prime). Therefore, with the exception of $x = 4$, $3|x$ if $x - 1$ and $x + 1$ are prime. Since $2|x$ and $3|x$, $6|x$ when $x - 1$ and $x + 1$ are prime, with the one exception already mentioned.

Problem 5

Running time of the Euclidean algorithm. Suppose $a > b > 0$. Set $r_0 = a$, $r_1 = b$ and run the Euclidean algorithm as follows:

$$r_0 = q_1r_1 + r_2 \quad (0 \leq r_2 < r_1)$$
$$r_1 = q_2r_2 + r_3 \quad (0 \leq r_3 < r_2)$$
$$r_2 = q_3r_3 + r_4 \quad (0 \leq r_4 < r_3)$$
$$\vdots$$
$$r_{k-2} = q_{k-1}r_{k-1} + r_k \quad (r_k = 0).$$

It stops at the last step because $r_k = 0$.

(i) Show for all $0 \leq j \leq k$ that $r_{k-j} \geq F_j$, where $F_j$ is the $j$-th Fibonacci number. Hint: induction.

(ii) Use the exact formula for $F_k$ to prove that the number of steps, $k$, required in the Euclidean algorithm satisfies

$$k \leq \frac{\log_{10}(a\sqrt{5} + 1)}{\log_{10}\phi}.$$
Here the logarithms are to base-10. For example, when $a = 10^{1000}$, the upper bound is about 4786.64.

**Solution**

(i) We use induction, but it is a *finite* induction. Since $r_k = 0$ and $r_{k-1} \geq 1$ (because the algorithm did not stop at the $k-1$ step), the base case holds. Now assume that $r_k \geq F_0, \ldots, r_{k-j} \geq F_j$, where $j$ is some number satisfying $1 \leq j \leq k - 1$. Since $q_{r-j} \geq 1$, it follows that

$$r_{k-(j+1)} = q_{r-j} r_{k-j} + r_{k-(j-1)} \geq r_{k-j} + r_{k-(j-1)} \geq F_j + F_{j-1} = F_{j+1}.$$ 

By induction, $r_{k-j} \geq F_j$ for all $0 \leq j \leq k$.

(ii) By part (i), $a \geq F_k$. Since $|\sqrt{5} - 1| < 1$, the exact formula for Fibonacci numbers gives

$$a \geq F_k \geq \frac{\phi^k - 1}{\sqrt{5}}.$$ 

Rearranging gives $a\sqrt{5} + 1 \geq \phi^k$. Taking logarithms yields the desired inequality.