Problem 1
The Multinomial Theorem.
(a) Prove that for any \(n \geq 0\),
\[
(x + y + z)^n = \sum_{a, b, c \geq 0} \frac{n!}{a!b!c!} x^a y^b z^c.
\]
Hint: Write \(x + y + z = (x + y) + z\) and use the binomial theorem.

(b) Prove that for any \(m \in \mathbb{N}\), any real numbers \(x_1, \ldots, x_m\) and any positive integer \(n\) that
\[
(x_1 + x_2 + \cdots + x_m)^n = \sum_{a_1, a_2, \ldots, a_m \geq 0} \frac{n!}{a_1!a_2! \cdots a_m!} x_1^{a_1} \cdots x_m^{a_m}.
\]
Hint: induction on \(m\).

Solution
(a) Apply the binomial theorem first using the hint, then apply it again:
\[
(x + (y + z))^n = \sum_{a=0}^{n} \binom{n}{a} x^n z^{n-a} = \sum_{a=0}^{n} \binom{n}{a} x^a z^{n-a}.
\]
Notice in the sum that the exponents satisfy \(a + b + (n-a-b) = n\). Change variables to \(c = n-a-b\), and notice that
\[
\binom{n}{a} \binom{n-a}{b} = \frac{n!}{a!b!(n-a-b)!}.
\]

(b) By induction on \(n\), assuming that it is true for some \(m \geq 2\) and for all real numbers \(x_1, \ldots, x_m\) and all positive integers \(n\). Now let \(x_1, \ldots, x_{m+1}\) be real numbers, and \(n \in \mathbb{N}\). Using the \(m\)-case (the inductive hypothesis) and the \(m = 2\) case (the binomial theorem) together we get
\[
(x_1 + x_2 + \cdots + x_{m+1})^n = (x_1 + (x_2 + \cdots + x_{m+1}))^n = \sum_{a_1=0}^{n} \binom{n}{a_1} x_1^a (x_2 + \cdots + x_{m+1})^{n-a}
\]
\[
= \sum_{a_1=0}^{n} \binom{n}{a_1} x_1^a \sum_{a_2, a_3, \ldots, a_m \geq 0} \frac{(n-a_1)!}{a_2! \cdots a_m!} x_2^{a_2} \cdots x_m^{a_m} x_{m+1}^{a_{m+1}}
\]
\[
= \sum_{a_1, a_2, \ldots, a_m+1 \geq 0} \frac{n!}{a_1!a_2! \cdots a_m!a_{m+1}!} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m} x_{m+1}^{a_{m+1}}.
\]
By induction, the claimed formula holds for every \(m\).
Problem 2

Problem 4.25 (c,d,e). are these functions surjections?

Solution

c) $f(a, b) = ab + 1/2$ — YES. For $n \in \mathbb{N}$, $f(n, 1) = n$, so $n$ is in the image.
d) $f(a, b) = (a + 1)b(b + 1)/2$ — NO. When $a, b \in \mathbb{N}$, $(a + 1)b(b + 1)/2 \geq 2$, so the image does not contain 1.
e) $f(a, b) = ab + a/2$ — NO. We have $f(1, 1) = 1$. When $\min \{a, b\} = 1$ and $\max \{a, b\} = 2$, we have $ab(a + 1)/2 \geq 3$. When $a \geq 2$ and $b \geq 2$, we have $ab(a + 1)/2 \geq 8$. Thus the image does not contain 2.

Problem 3

Problem 4.31. If $A, B \subseteq \mathbb{R}$, $f : A \rightarrow B$ is a bijection and increasing, then $f^{-1}$ is increasing.

Solution

We prove the contrapositive: if $f^{-1}$ is not increasing, then $f$ is not increasing. If $f^{-1}$ is not increasing, this means that for some numbers $b_1, b_2 \in B$ we have $b_1 < b_2$ and $f^{-1}(b_1) \geq f^{-1}(b_2)$. It is not possible that $f^{-1}(b_1) = f^{-1}(b_2)$ because $f^{-1}$ is a bijection, therefore $f^{-1}(b_1) > f^{-1}(b_2)$. Let $a_1 = f^{-1}(b_1)$ and $a_2 = f^{-1}(b_2)$. Then $a_1 > a_2$ and $f(a_1) < f(a_2)$, which proves that $f$ is not increasing.

Problem 4

Problem 4.34 (a,b,c,d). Properties of $h = g \circ f$, where $f : A \rightarrow B$, $g : B \rightarrow C$.

Solution

(a) If $h$ is injective, then $f$ is injective. True. Suppose $h$ is injective and $f$ is not injective. Then there are different elements $a_1 \in A$, $a_2 \in A$ such that $f(a_1) = f(a_2) = b$, where $b \in B$. Then $h(a_1) = g(f(a_1)) = g(b) = g(f(a_2)) = h(a_2)$, contradicting the injectivity of $f$.

(b) If $h$ is injective, then $g$ is injective. False. Let $A = [0, \infty)$, $B = C = \mathbb{R}$, $f(x) = x$, $g(x) = x^2$.

(c) If $h$ is surjective, then $f$ is surjective. False. Let $A = B = \mathbb{R}$, $C = [0, \infty)$, $f(x) = x^2$, $g(x) = x^2$.

(d) If $h$ is surjective, then $g$ is surjective. True. Since $h$ is surjective, for every $c \in C$ there is an $a \in A$ with $h(a) = c$.

Problem 5

Prove that for any positive integer $n$, $\mathbb{Z}^n$ is countable.

Solution

This can be done by induction on $n$, or directly. Here is a direct proof. For $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, define $g(k_1, \ldots, k_n) = \max(|k_1|, \ldots, |k_n|)$. There are finitely many $n$-tuples with a given value of $g(k_1, \ldots, k_n) = m$, so we can list the tuples with $g(k_1, \ldots, k_n) = 0$ (a single n-tuple), then those with $g(k_1, \ldots, k_n) = 1$, then those with $g(k_1, \ldots, k_n) = 2$, etc. Define $f : \mathbb{N} \rightarrow \mathbb{Z}^n$ so that $f(n)$ is the $n$-th term in the list.

An induction proof proceeds by writing $\mathbb{Z}^n = \mathbb{Z}^{n-1} \times \mathbb{Z}$, and following the method of diagonals that proves that $\mathbb{N}^2$ is countable.