Problem 1
Problem 3.3. $P(n)$ implies both $P(n - 1)$ and $P(n + 1)$.

Solution
Let $Q(n)$ be the statement “$P(n)$ and $P(-n)$ are both true.” We prove that $Q(0), Q(1), \ldots$ are all true by induction. $Q(0)$ is given to be true. Assuming that $Q(n)$ is true, we know that $P(n)$ and $P(-n)$ are both true. But $P(n)$ implies $P(n + 1)$, and $P(-n)$ implies $P(-n - 1)$. Hence $Q(n + 1)$ is true. By induction, $Q(n)$ is true for every $n$. Therefore, $P(n)$ is true for all $n \in \mathbb{Z}$. (This is a kind of “two-sided” induction)

Problem 2
Problem 3.49 (b). determine the $n \in \mathbb{N}$ so that $2^n \geq (n + 1)^2$.

Solution
The inequality holds for all $n \geq 6$. By direct calculation, the inequality fails for $1 \leq n \leq 5$. Let $P(n)$ be the statement $2^n \geq (n + 1)^2$. $P(6)$ is true because $64 \geq 49$. Assume $k \geq 6$ and $P(k)$ is true, that is, $2^k \geq (k + 1)^2$. Doubling both sides of the inequality yields $2^{k+1} \geq 2(k + 1)^2$. We want the right side to be $\geq (k + 2)^2$. Since $k \geq 6$, $k^2 \geq 36$, so

$$2^{k+1} \geq 2(k + 1)^2 = k^2 + 4k + 2 + k^2 \geq k^2 + 4k + 4 = (k + 2)^2,$$

which proves $P(k + 1)$.

Problem 3
Problem 3.56/ $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$.

Solution
(a) If $a_1$ and $a_2$ are odd, then $a_n$ is odd for all $n \in \mathbb{N}$. Proof by induction on $n$. For the base case, we are given that $a_1$ and $a_2$ are odd. Suppose $k \geq 2$ and assume $a_1, \ldots, a_k$ are all odd. In particular, $a_k$ and $a_{k-1}$ are odd. By the recurrence formula, $a_{k+1} = 2a_k + 3a_{k-1}$. Always, $2a_k$ is even. Since $a_{k-1}$ is odd, $3a_{k-1}$ is odd. An even number plus an odd number is odd, so $a_{k+1}$ is odd. By induction, $a_n$ is odd for all $n \in \mathbb{N}$.

(b) Prove that $a_n = \frac{1}{2}(3^{n-1} - (-1)^n)$ for all $n$. The formula holds for $n = 1$ and $n = 2$, and these two cases form the base step. Assume that $k \geq 2$ and that the formula is true for $1 \leq n \leq k$. In particular, the formula is true for $n = k$ and $n = k - 1$. Then

$$a_{k+1} = 2a_k + 3a_{k-1} = \frac{1}{2}(3^{k-1} - (-1)^k) + \frac{3}{2}(3^{k-2} - (-1)^{k-1}) = \frac{1}{2}(3^k - (-1)^{k+1}),$$

since $2(-1)^k + 3(-1)^{k-1} = (-1)^{k-1}(-2 + 3) = (-1)^{k-1} = (-1)^{k+1}$. Thus, the formula is true for $n = k$. By induction, the formula is true for all $n \in \mathbb{N}$. 
Problem 4

Let \( F_k \) denote the \( k \)-th Fibonacci number: \( F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \).

(a) Prove that \( 2 | F_{3k} \) for every \( k \geq 0 \).

(b) Prove that \( 3 | F_{4k} \) for every \( k \geq 0 \).

Solution

(a) Since \( F_0 = 0 \), the statement is true for \( k = 0 \). Now suppose that the statement is true for \( n = k, \) where \( k \geq 0 \). That is, \( 2 | F_{3k} \). We want to express \( F_{3(k+1)} \) in terms of \( F_{3k} \), which we can do by repeated application of the Fibonacci recurrence:

\[
F_{3k+3} = F_{3k+2} + F_{3k+1} = 2F_{3k+1} + F_{3k}.
\]

Since \( 2 | F_{3k} \), it follows that \( 2 | 2F_{3k+1} + F_{3k} \) also, which means \( 2 | F_{3(k+1)} \). By induction, \( 2 | F_{3k} \) for every \( k \geq 0 \).

(b) The statement is true when \( n = 0 \), as \( F_0 = 0 \). Now suppose that the statement is true for \( n = k, \) where \( k \geq 0 \). (We can use simple induction here). We want to express \( F_{4(k+1)} \) in terms of \( F_{4k} \), which we can do by repeated application of the Fibonacci recurrence:

\[
F_{4k+4} = F_{4k+3} + F_{4k+2} = (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k})
= F_{4k+2} + 2F_{4k+1} + F_{4k} = (F_{4k+1} + F_{4k}) + 2F_{4k+1} + F_{4k}
= 3F_{4k+1} + 2F_{4k}.
\]

Since \( 3 | F_{4k} \) by hypothesis and \( 3 | 3F_{4k+1} + 2F_{4k} \), we conclude that \( 3 | F_{4(k+1)} \) and completes the induction step. By induction, \( 3 | F_{4n} \) for all \( n \geq 0 \).

Problem 5

Suppose you have an unlimited number of 5-cent and 7-cent stamps. Show that for any integer \( k \geq 24 \), you can make exactly \( k \) cents postage. Hint: If \( P(k) \) is the statement “it is possible to make \( k \) cents postage with 5-cent and 7-cent stamps”, prove that \( P(k) \implies P(k + 5) \).

Solution

First, check that 24, 25, 26, 27 and 28 cents postage is possible. Now assume that \( n \geq 28 \) and all postage amounts from 24 to \( n \) can be made. That is, for every \( k \) from 24 to \( n \), there are non-negative integers \( x, y \) so that \( k = 5x + 7y \). Consider \( n + 1 \). We know that \( n + 1 - 5 \geq 24 \), hence \( n + 1 - 5 \) cents postage is possible. By adding one 5-cent stamp, we make \( n + 1 \) cents postage. Therefore, \( P(n + 1) \) is true. By induction, all postage amounts \( n \geq 24 \) are possible.