Problem 1
(13.30) Let \( a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \) for \( n \in \mathbb{N} \).

(a) Show that the sequence \( (a_n) \) is bounded.

(b) Show that \( (a_n) \) is monotone. Conclude that \( \lim_{n \to \infty} a_n \) exists.

(c) (BONUS) Find \( \lim_{n \to \infty} a_n \), with proof.

Solution
(a) for every \( n \),
\[
a_n < \frac{1}{n} + \cdots + \frac{1}{n} = 1 \quad (n \text{ summands}).
\]
Thus, \( (a_n) \) is bounded above by 1. Clearly, \( a_n > 0 \) for all \( n \).

(b) We compute
\[
a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0.
\]
Thus, \( (a_n) \) is monotone increasing.

(c) the limit is \( \ln 2 \). Let \( f(t) = \frac{1}{t} \). Then
\[
a_n = \frac{1}{n} (f(1+1/n) + f(1+2/n) + \cdots + f(2)).
\]
This is a Riemann sum for \( \int_1^2 f(t) \, dt \), with \( n \) intervals, the approximation using the right-endpoint rule. Therefore,
\[
\lim_{n \to \infty} a_n = \int_1^2 \frac{dt}{t} = \ln 2.
\]

Problem 2
Problem 14.19.

Solution
Starting with \( x_2 \), the sequence is monotone decreasing and bounded below by \( \sqrt{3} \). Proof: suppose that \( \sqrt{3} \leq x_n \). Then
\[
x_{n+1} - x_n = \frac{3/x_n - x_n}{2} = \frac{3 - x_n^2}{2x_n} \leq 0
\]
and also
\[
x_{n+1}^2 = \left( \frac{x_n + 3/x_n}{2} \right)^2 = \frac{x_n^2 + 6 + (3/x_n)^2}{4} = \frac{x_n - 3/x_n}{2}^2 + 3 \geq 3.
\]
So we get the desired property by induction. Thus, by the Monotone Convergence Theorem, the limit \( L \) exists and \( L \geq \sqrt{3} \). By Limit Theorems, \( L = \frac{L + 3/L}{2} \), which implies \( L^2 = 3 \), \( L = \pm \sqrt{3} \), so \( L = \sqrt{3} \) since all terms are positive.
Problem 3

Problem 14.20. Suppose that \( x_1 > -1 \) and that \( x_{n+1} = \sqrt{1 + x_n} \) for \( n \geq 1 \). Prove that \( \lim x_n \) exists and find the limit.

Solution

First, show by induction that \( 0 < x_n \) for all \( n \geq 2 \) (very easy). By limit theorems, if \( L = \lim x_n \) exists, then \( L^2 = 1 + L \), which implies that \( L = \frac{1+\sqrt{5}}{2} \). Because \( x_n > 0 \) for all \( n \geq 2 \), \( L = \frac{1+\sqrt{5}}{2} \) is the only possible limit.

If \( x_1 \leq \frac{1+\sqrt{5}}{2} \), then by induction it is easy to show \( x_n \leq x_{n+1} \leq \frac{1+\sqrt{5}}{2} \) for all \( n \); hence by the Monotone Convergence Theorem, \( \lim x_n \) exists. Therefore, \( \lim x_n = L = \frac{1+\sqrt{5}}{2} \).

If \( x_1 > \frac{1+\sqrt{5}}{2} \), then by induction it is easy to show \( x_n > x_{n+1} > \frac{1+\sqrt{5}}{2} \) for all \( n \); hence by the Monotone Convergence Theorem, \( \lim x_n \) exists. Therefore, \( \lim x_n = L = \frac{1+\sqrt{5}}{2} \).

Problem 4

Problem 14.30. \( x_1 = 1 \) and \( x_{n+1} = 1/(x_1 + \cdots + x_n) \). Prove that \( \lim x_n \) exists and find the limit.

Solution

First, \( 0 < x_{n+1} \leq x_n \) for all \( n \), which is easy to prove by induction. Indeed, \( x_2 = 1/x_1 = 1 \) satisfies this inequality for \( n = 1 \). If \( 0 < x_{m+1} \leq x_m \) for \( 1 \leq m \leq n-1 \), then \( x_{n+1} = 1/(x_1 + \cdots + x_n) \) is clearly positive and less than \( x_n = 1/(x_1 + \cdots + x_{n-1}) \). By induction, \( 0 < x_{n+1} \leq x_n \) for all \( n \), so \( (x_n) \) is monotone non-increasing and bounded below by zero. By the Monotone Convergence Theorem, the limit \( L \) exists and \( L \geq 0 \).

Assume that \( L > 0 \). Then by properties of monotone sequences, \( x_n \geq L \) for every \( n \), and therefore for each \( n \geq 2 \),

\[
x_n = \frac{1}{x_1 + \cdots + x_{n-1}} \leq \frac{1}{L + \cdots + L} = \frac{1}{(n-1)L}.
\]

So we have \( 0 < x_n \leq \frac{1}{(n-1)L} \) for \( n \geq 2 \), and by the Squeeze Law, \( \lim x_n = 0 = L \), a contradiction. Therefore, \( L = 0 \).

Alternatively, suppose \( \lim x_n = L \) exists and is not zero. We have

\[
x_n = (1 + \cdots + x_{n-1} + x_n) - (1 + \cdots + x_{n-1}) = \frac{1}{x_{n+1}} - \frac{1}{x_n}.
\]

By the limit laws, \( L = \frac{1}{L} - \frac{1}{L} = 0 \), a contradiction. So the only possible limit is zero.

Problem 5

Problem 14.42, variation. Measure zero. A set \( S \subset \mathbb{R} \) has measure zero is, for every \( \varepsilon > 0 \), there is a countable set of open intervals \( I_1, I_2, \ldots \) which contain \( S \) such that the sum of the lengths of the intervals is less than \( \varepsilon \). In plain language, the set \( S \) “takes up virtually no space”.

(a) Show that any countable subset of \( \mathbb{R} \) has measure zero. In particular, \( \mathbb{Q} \) has measure zero. Hint: consider \( \varepsilon /2^k \) for \( k = 1, 2, \ldots \) (Contrast this with the fact that \( \mathbb{Q} \) is dense in \( \mathbb{R} \).)

(b) Show that the union of countable many sets of measure zero also has measure zero.
Solution

(a) Let $A = \{a_1, a_2, \ldots\}$ be a countable set. Each element $a_j$ is clearly inside the interval $I_j = (a_j - \varepsilon/2^{j+1}, a_j + \varepsilon/2^{j+1})$. The intervals $I_1 \cup I_2 \cup \cdots$ include all of $A$, and have total length $\varepsilon/2$.

(b) Let $A_1, A_2, \ldots$ be sets with measure zero, and let $\varepsilon > 0$. For each $i$, there is a countable collection of open intervals $I_{i,1}, I_{i,2}, \ldots$ containing $A_i$ and with the sum of the lengths of the intervals less than $\varepsilon/2^i$. Then $A_1 \cup A_2 \cup \ldots$ is covered by all of the sets $I_{i,j}$, which have total length less then $\varepsilon$. Moreover, the set of intervals $I_{i,j}$ is countable, as the set of indices is $\mathbb{N}^2$, which is a countable set.

Problem 6

(Problem 14.53, plus) **Conditionally convergent series.** (15 points) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots.$$ By Problem 3, the series converges but the series of absolute values of terms, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (you may take this as a fact; no need to prove it). The series is called *conditionally convergent*, and you may have wondered why the name.

(a) Show that the series converges to a sum less than 1 (should be very short and easy).

(b) Consider the following series with the same terms but rearranged:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{4} + \cdots,$$

where the pattern is four positive terms, then one negative term. Prove that the sum of this series is larger than 1 (you may assume that the limit exists).

(c) Find a rearrangement of the terms of the series that has sum greater than 100, and a rearrangement of the terms of the series that has sum less than -100.

Solution

(a) By the analysis of the alternating series in the previous problem, the first partial sum $a_1 = 1$ is an upper bound for the series (the odd partial sums are decreasing).

(b) Let $s_n$ be the $n$-th partial sum (sum of the first $n$ terms). We claim that $s_{5k} > 1$ for all $k$. This is true for $k = 1$ by checking. If true for a particular $k$, then

$$s_{5(k+1)} = s_{5k} + \frac{1}{8k+1} + \frac{1}{8k+3} + \frac{1}{8k+5} + \frac{1}{8k+7} - \frac{1}{2k+2} > s_{5k} + \frac{4}{8k+8} - \frac{1}{2k+2} = s_{5k} + 1.$$

By induction, $s_{5k} > 1$ for all $k$. Assuming the limit exists, the subsequence $s_{5k}$ exists and has the same limit, hence this limit is larger than 1.

(c) The main point is that the positive terms sum to $\infty$ and the negative terms sum to $\infty$, and $\infty - \infty$ is an “indeterminate form”. To get a sum more than 100, take an initial sequence of positive terms $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k+1}$ whose sum is larger than 101. Then take $-\frac{1}{2}$. Next take more positive terms $\frac{1}{2k+3} + \cdots + \frac{1}{2m+1}$ whose sum is larger than $\frac{1}{2}$ (possible because the tail of the series starting at $\frac{1}{2k+3}$ diverges). Then take $-\frac{1}{4}$. Then take positive terms $\frac{1}{2m+3} + \cdots + \frac{1}{2n+1}$ with sum larger than $\frac{1}{4}$, etc. The construction for a sum less than -100 is similar, by first taking negative summands $-\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2m}$ that sum to less than -102, then adding 1, then more negative summands, etc.