Integers with a divisor in a given interval

Kevin Ford March 2006

Define $\tau(n,y,z) = |\{d|n : y < d \le z\}|$.

By inclusion-exclusion,

$$H(x,y,z) := |\{n \le x : \tau(n,y,z) \ge 1\}| = \sum_{k\ge 1}(-1)^{k-1} \sum_{y<d_1<\cdots<d_k\le z} \left\lfloor \frac{x}{\mathrm{lcm}[d_1,\cdots,d_k]} \right\rfloor.$$

The density of integers with a divisor in $(y,z]$ is

$$\varepsilon(y,z) = \lim_{x\to\infty} \frac{H(x,y,z)}{x}, = \sum_{k\ge 1}(-1)^{k-1} \sum_{y<d_1<\cdots<d_k\le z} \frac{1}{\mathrm{lcm}[d_1,\cdots,d_k]}.$$

Besicovitch (1934) : $\liminf_{y\to\infty} \varepsilon(y,2y) = 0$

Erdős (1935) : $\lim_{y\to\infty} \varepsilon(y,2y) = 0$

Erdős (1960) : $\varepsilon(y,2y) = (\log y)^{-\delta+o(1)}$,

$$\delta = 1 - \tfrac{\log(e\log 2)}{\log 2} = 0.08607\ldots.$$

$H(x,y,z)$ when $z-y$ is small.

Let $z = e^\eta y$, $0 < \eta \le 1$ and $y \le \sqrt{x}$. Then

$$H(x,y,z) = \sum_{y<d\le z} \left\lfloor \frac{x}{d} \right\rfloor + O\left(\sum_{y<d_1<d_2\le z} \frac{x}{\mathrm{lcm}[d_1,d_2]} \right)$$

$$= x\left(\sum_{y<d\le z} \frac{1}{d} + O\left(\sum_{y<d_1<d_2\le z} \frac{1}{\mathrm{lcm}[d_1,d_2]} \right) \right) + O(z-y)$$

$$= x \sum_{y<d\le z} \frac{1}{d} + O\left(x \sum_{m\le \eta y} \frac{1}{m} \sum_{\frac{y}{m}<t_1<t_2\le\frac{z}{m}} \frac{1}{t_1 t_2} \right) + O(\eta y)$$

$$= x \sum_{y<d\le z} \frac{1}{d} + O(\eta^2 x \log y).$$

Therefore, if $z-y\to\infty$ and $z-y = o(y/\log y)$, then

$$H(x,y,z) \sim \eta x.$$

Tenenbaum, 1984 : $H(x,y,z)\sim \eta x$ for $z-y\to\infty$

and $z \lesssim z_0(y)$, where

$$z_0(y) = y + \frac{y}{(\log y)^{\log 4 - 1}}.$$

$H(x, y, z)$ when z is large.

Let $y^2 \leq z \leq \sqrt{x}$. By sieve methods, the number of $n \leq x$ that do not have a *prime* divisor in $(y, z]$ is

$$\ll x \prod_{y<p\leq z} \left(1 - \frac{1}{p}\right) \ll x \frac{\log y}{\log z}.$$

Thus, if $z \leq \sqrt{x}$ and $\frac{\log z}{\log y} \to \infty$, then

$$H(x, y, z) \sim x.$$

$H(x, y, z)$ for intermediate z.

Analysis much more difficult. Tenenbaum in 1984 gave reasonably sharp bounds when $z_0(y) \leq z \leq y^2$, e.g.

$$e^{-c\sqrt{\log\log y \log\log\log y}} \ll \frac{H(x, y, 2y)(\log y)^\delta}{x} \ll \frac{1}{\sqrt{\log\log y}},$$

where $\delta = 1 - \frac{1+\log\log 2}{\log 2} = 0.08607\ldots$.

New results (2004)

$$z = e^\eta y = y^{1+u}, \quad \eta = (\log y)^{-\beta},$$

$$\beta = \log 4 - 1 + \tfrac{\xi}{\sqrt{\log\log y}},$$

$$z_0(y) = y + \frac{y}{(\log y)^{\log 4 - 1}},$$

$$G(\beta) = \tfrac{1+\beta}{\log 2} \log\left(\tfrac{1+\beta}{e\log 2}\right) + 1 \quad (0 \le \beta \le \log 4 - 1)$$

Theorem 1. *Uniformly in* $100 \le y \le \sqrt{x}$, $z \ge y+1$,

$$\frac{H(x,y,z)}{x} \asymp \begin{cases} \eta = \log(z/y) & y+1 \le z \le z_0(y) \\ \frac{\beta}{(1-\xi)(\log y)^{G(\beta)}} & z_0(y) \le z \le 2y \\ u^\delta (\log \frac{2}{u})^{-3/2} & 2y \le z \le y^2 \\ 1 & z \ge y^2. \end{cases}$$

Corollary.

$$H(x,y,2y) \asymp \frac{x}{(\log y)^\delta (\log\log y)^{3/2}}.$$

Short interval version

Theorem 2. *For* $y_0 \le y \le \sqrt{x}$*,* $z \ge y+1$ *and and* $\frac{x}{\log^{10} z} \le \Delta \le x$*, we have*

$$H(x,y,z) - H(x-\Delta,y,z) \asymp \frac{\Delta}{x} H(x,y,z).$$

Here y_0 is a large constant.

Squarefree integers

Let $H^*(x,y,z)$ be the number of *squarefree* numbers $n \le x$ with $\tau(n,y,z) \ge 1$.

Theorem 3. *Suppose* $y_0 \le y \le \sqrt{x}$*,* $y+1 \le z \le x$ *and* $\frac{x}{\log y} \le \Delta \le x$*. If* $z \ge y + Ky^{1/5}\log y$*, where* K *is a large absolute constant, then*

$$H^*(x,y,z) - H^*(x-\Delta,y,z) \asymp \frac{\Delta}{x} H(x,y,z).$$

Some applications (using $z \asymp y$ case)

1. (Erdős 1955/60, Linnik, A. I. Vinogradov). Let $A(x) = |\{n = m_1 m_2 : m_i \le \sqrt{x}\}|$.

Corollary A. $A(x) \asymp \dfrac{x}{(\log x)^{\delta}(\log\log x)^{3/2}}$.

2. Distribution of Farey gaps (Cobeli, Ford, Zaharescu, 2003). Farey fractions order Q: $\frac{0}{1}, \frac{1}{Q}, \frac{1}{Q-1}, \cdots, \frac{Q-1}{Q}, \frac{1}{1}$.

Corollary B. # of distinct gaps in the sequence is

$$\asymp \frac{Q^2}{(\log Q)^{\delta}(\log\log Q)^{3/2}}.$$

3. Erdős function

$$\tau^+(n) = |\{k \in \mathbb{Z} : \tau(n, 2^k, 2^{k+1}) \ge 1\}|.$$

Corollary C.

$$\frac{1}{x}\sum_{n\le x} \tau^+(n) \asymp \frac{(\log x)^{1-\delta}}{(\log\log x)^{3/2}}.$$

Divisors of shifted primes

$H(x, y, z; \mathscr{A}) = |\{n \le x : n \in \mathscr{A}, \tau(n, y, z) \ge 1\}$ for a set $\mathscr{A} \subseteq \mathbb{N}$. Fix $\lambda \ne 0$, let $P_\lambda = \{p + \lambda : p \text{ prime}\}$.

Theorem 4. *Let λ be a fixed non-zero integer. Let $1 \le y \le \sqrt{x}$ and $y + 1 \le z \le x$. Then*

$$H(x, y, z; P_\lambda) \ll_\lambda \begin{cases} \dfrac{H(x, y, z)}{\log x} & z \ge y + (\log y)^{2/3} \\[2ex] \dfrac{x}{\log x} \displaystyle\sum_{y < d \le z} \dfrac{1}{\phi(d)} & y < z \le y + (\log y)^{2/3}. \end{cases}$$

Theorem 5. *For fixed λ, a, b with $\lambda \ne 0$ and $0 \le a < b \le 1$, we have*

$$H(x, x^a, x^b; P_\lambda) \gg_{a,b,\lambda} \frac{x}{\log x}.$$

Integers with exactly r divisors in $(y, z]$

$$H_r(x, y, z) = |\{n \le x : \tau(n, y, z) = r\}|,$$
$$\varepsilon_r(y, z) = \lim_{x\to\infty} \frac{H_r(x, y, z)}{x}.$$

Erdős conjecture, 1960: $\lim_{y\to\infty} \frac{\varepsilon_1(y, 2y)}{\varepsilon(y, 2y)} = 0.$

Tenenbaum conjectures, 1987:

1. $\forall r \ge 1$, $\liminf_{y\to\infty} \frac{\varepsilon_r(y, 2y)}{\varepsilon(y, 2y)} > 0.$
2. $\forall r \ge 1$, if $z/y \to \infty$, then $\lim_{y\to\infty} \frac{\varepsilon_r(y, z)}{\varepsilon(y, z)} = 0.$

Hall & Tenenbaum conjecture, 1988:

$$\lim_{y\to\infty} \frac{\varepsilon_r(y, 2y)}{\varepsilon(y, 2y)} = d_r > 0.$$

Tenenbaum's 1987 paper

I. When $y + \frac{y}{(\log y)^{\log 4 - 1}} < z \le 2y$,

$$\frac{1}{Z(y)} \ll_r \frac{\varepsilon_r(y,z)}{\varepsilon(y,z)} \le 1.$$

II. When $2y \le z \le y^2$,

$$\frac{1}{Z(y)\log(z/y)} \ll_r \frac{\varepsilon_r(y,z)}{\varepsilon(y,z)} \ll_r \frac{Z(y)}{\log(z/y)^{\delta}},$$

Here $Z(y) = \exp\{c_r\sqrt{\log\log y \log\log\log y}\}$.

New bounds for $H_r(x,y,z)$ (2004)

Theorem 6. *If $z \geq y+1$, then*

$$\frac{\varepsilon_1(y,z)}{\varepsilon(y,z)} \asymp \frac{\log\log(z/y+5)}{\log(z/y+5)}.$$

Theorem 7. *Suppose $r \geq 2$, $b > 0$ fixed and small, $C > 1$ fixed and large, and $y + \frac{y}{(\log y)^{\log 4 - 1 - b}} \leq z \leq y^C$,*

$$\frac{\varepsilon_r(y,z)}{\varepsilon(y,z)} \asymp_{r,b,C} \frac{(\log\log(z/y+5))^{\nu(r)+1}}{\log(z/y+5)},$$

where $2^{\nu(r)} \| r$.

Corollary D. $\forall r \geq 1$, $\forall c > 1$, if $y \geq y_0(r)$, then

$$\frac{\varepsilon_r(y,cy)}{\varepsilon(y,cy)} \gg_{r,c} 1.$$

Corollary E. If $z/y \to \infty$, then

$$\frac{\varepsilon_r(y,z)}{\varepsilon(y,z)} \to 0.$$

Further bounds for $H_r(x,y,z)$

Theorem 8 (Ford, Tenenbaum, 2006)**.** *If* $y \le \sqrt{x}$ *and* $y < z \le y + \frac{y}{(\log y)^{\log 4 - 1 - o(1)}}$, *then*

$$\sum_{r \ge 2} H_r(x,y,z) = o(H(x,y,z)).$$

Theorem 9 (Yong Hu, 2006)**.** *If* $y^{20} \le z \le x^{1/4}$, *then*

$$\frac{H_2(x,y,z)}{x} \asymp \frac{\log\log y \log\log z}{\log z}$$

and

$$\frac{H_3(x,y,z)}{x} \asymp \frac{\log\log z + (\log\log z - \log\log y)^2}{\log z}.$$

Conjecture: Under the hypotheses of Theorem 9, for each r,

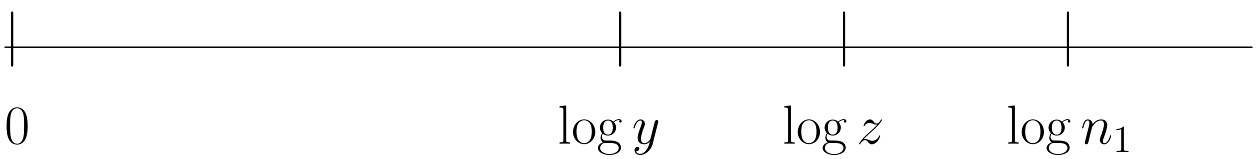
$$\frac{H_r(x,y,z)}{x} \asymp_r \frac{Q_r(\log\log y, \log\log z)}{\log z}$$

for some polynomial Q_r.

Proof ideas: $H(x, y, 2y)$

Suppose n is squarefree, and $n_1 := \prod_{p|n, p \le z} p \le z^{10}$.

Assume $\{\log d : d|n_1\}$ **is roughly uniformly distributed in** $[0, \log n_1]$**.**



If $\omega(n_1) = |\{p : p|n_1\}| = k$, then

$$\mathrm{Prob}[\tau(n_1, y, z) = 1] \approx \frac{\log(z/y)}{\log n_1} 2^k \approx \frac{2^k}{\log z}.$$

With $k_0 = \left\lfloor \frac{\log\log z}{\log 2} \right\rfloor$, we predict that

$$\begin{aligned} H(x, y, 2y) &\approx \sum_{k \ge k_0} |\{n \le x : \omega(n_1) = k\}| \\ &\approx \sum_{k \ge k_0} \frac{x (\log\log z)^{k-1}}{(k-1)! \log z} \\ &\asymp \frac{x (\log\log y)^{k_0}}{k_0! \log y} \\ &\asymp \frac{x}{(\log y)^{\delta} (\log\log y)^{1/2}}. \end{aligned}$$

Two principles

Principle # 1. The numbers $\{\log\log p : p|n_1\}$ behave like uniformly distributed random numbers in $[0, \log\log z]$:

$$\log\log p_j(n_1) \approx \frac{j \log\log z}{k} \quad (1 \le j \le k),$$

where $p_j(n_1)$ is the j-th smallest prime factor of n_1.

Principle # 2. The numbers $\{\log d : d|n_1\}$ **do not** behave like uniformly distributed random numbers in $[0, \log n_1]$.

Principle #1 $\implies$ **Principle # 2.** Reason: with high probability,

$$\log\log p_j(n_1) \le \frac{j \log\log z}{k} - c\sqrt{\log\log z}$$

for some j. This causes the numbers $\{\log d : d|n_1\}$ to be grouped in isolated clumps. In fact, for most n_1,

$$\mathrm{Prob}[\tau(n_1, y, 2y) \ge 1] \approx \frac{2^k}{\log z} \exp\{-c\sqrt{\log\log z}\}.$$

New ideas

Focus on **abnormal** integers, those satisfying

$$(1)\quad \log\log p_j(n_1) \ge \frac{j \log\log z}{k} - O(1) \quad (1 \le j \le k).$$

On average over n_1 satisfying (1), $\{\log d : d|n_1\}$ is distributed uniformly in $[0, \log n_1]$.

The probability that (1) occurs is $\asymp \frac{1}{k} \asymp \frac{1}{\log\log y}$. This leads to a refined prediction

$$H(x, y, 2y) \asymp \frac{x}{(\log y)^{\delta}(\log\log y)^{3/2}},$$

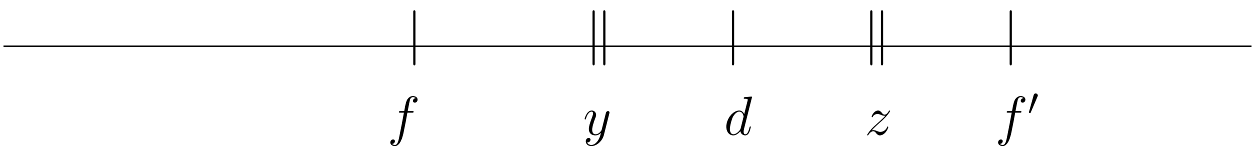
which is the correct order. More generally, we need to estimate the liklihood that

$$(2)\qquad \log\log p_j(n_1) \ge \frac{j \log\log z}{v} - u \quad (1 \le j \le k)$$

when v is close to k and $1 \le u \ll \sqrt{\log\log z}$.

Lower bound for $H_1(x, y, 2y)$

Say $d|n_1$ is **η-isolated** if there is no $f|n_1$ with $0 < |\log f/d| \le \eta$. Let $I(m;\eta)$ be the number of η-isolated $d|m$. If d is $\log 2$-isolated and $y < d < 2y$, then $\tau(n, y, 2y = 1)$.



The probability that $\tau(n_1, y, z) = 1$ is about $\dfrac{I(n_1; \log 2)}{\log z}$.
If $\omega(n_1) = k_0 - c_1$ and

$$\log\log p_j(n_1) \ge \frac{j \log\log z}{k} - c_2 \quad (1 \le j \le k),$$

then (on average) $I(n_1)$ will be close to $2^{k_0} \approx \log z$, i.e. no clumps means lots of isolated divisors. This leads to the prediction that

$$H_1(x, y, 2y) \gg \frac{x}{(\log y)^{\delta}(\log\log y)^{3/2}} \gg H(x, y, 2y).$$

Bounding $H_r(x, y, z)$ for $10y \le z \le y^{1.1}$

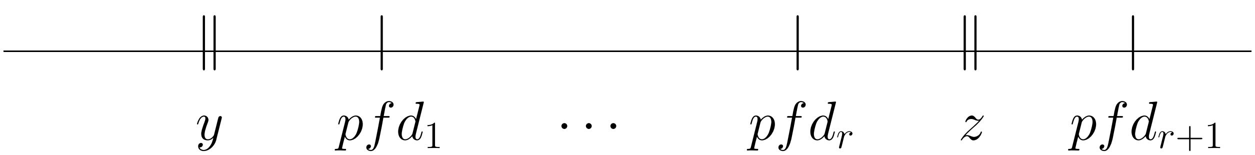
Recall

$$\frac{\varepsilon_r(y,z)}{\varepsilon(y,z)} \asymp_r \frac{(\log\log(z/y+5))^{\nu(r)+1}}{\log(z/y+5)},\ 2^{\nu(r)} \| r.$$

For lower bound, set $\eta = \log(z/y)$ and consider $n = pqst$,

$$s \le z/y,\ \tau(s) \ge r+1, \quad t < y^{1/4}, P^-(t) > z/y,$$

p is prime and $P^-(q) > z$. Let f be a 2η-isolated divisor of t, let $1 = d_1 < d_2 < \cdots < d_m = s$ be the divisors of s, and $pfd_r \le z < pfd_{r+1}$. Then $\tau(n, y, z) = r$.



Given pst, number of q is $\asymp \frac{x}{pst \log z}$. Then

$$\sum_{z/fd_r < p \le z/fd_{r+1}} \frac{1}{p} \asymp \frac{\log(d_{r+1}/d_r)}{\log z}.$$

Therefore,

$$H(x,y,z) \gg \frac{x}{\log^2 z} \sum_{\substack{t<y^{1/4} \\ P^-(t)>z/y}} \frac{I(t;2\eta)}{t} \sum_{\substack{s<z/y \\ \tau(s)\geq r+1}} \frac{\log(d_{r+1}/d_r)}{s}.$$

Write

$$s = p_1 \cdots p_k, \quad p_1 < \cdots p_k.$$

If $p_j > p_1 \cdots p_{j-1}$ for each j, then

$$\begin{array}{ll} d_1 = 1 & d_7 = p_3 p_2 \\ d_2 = p_1 & d_8 = p_3 p_2 p_1 \\ d_3 = p_2 & d_9 = p_4 \\ d_4 = p_2 p_1 & d_{10} = p_4 p_1 \\ d_5 = p_3 & d_{11} = p_4 p_2 \\ d_6 = p_3 p_1 & d_{12} = p_4 p_2 p_1, \text{ etc.} \end{array}$$

i.e., the ordinary ordering of divisors of s coincides with the lexographic ordering of the divisors. Then

$$\frac{d_{r+1}}{d_r} = \frac{p_t}{p_1 \cdots p_{t-1}}, \quad t = \nu(r) + 1.$$

Take s so that

$$p_j > (p_1 \cdots p_{j-1})^2 \qquad (1 \le j \le r),$$

then $\log(d_{r+1}/d_r) \gg \log p_t$, $t = \nu(r) + 1$. We obtain

$$\sum_s \frac{\log(d_{r+1}/d_r)}{s} \gg (\log z/y)(\log\log z/y)^t.$$

Upper bounds:

For simplicity assume s is squarefree.

Lemma 1. *Let* $s = p_1 \cdots p_k$, $p_1 < \cdots < p_k$. *For* $1 \le r \le 2^k - 1$, $\dfrac{d_{r+1}(n)}{d_r(n)} \le p_{\nu(r)+1}$.

With lemma 1, we obtain

$$\sum_s \frac{\log(d_{r+1}/d_r)}{s} \ll (\log z/y)(\log\log z/y)^{\nu(r)+1}.$$

Open problems/future projects

(I) Asymptotic formulas for $H(x, y, 2y)$ and $A(x)$.

(II) Lower bounds for $H(x, y, z; P_\lambda)$ when $z = y^{1+o(1)}$.

(III) Triple and higher order factorizations, e.g.

$$H(x, y_1, z_1, y_2, z_2) = |\{n \le x : d_1 d_2 | n, y_i < d_i \le z_i\}|,$$

$$A_3(x) = |\{n = m_1 m_2 m_3 : \text{ each } m_i \le x^{1/3}\}|.$$

(IV) Analogs for algebraic integers, e.g. count the Gaussian integers with norm $\le x$ and with a divisor in a given region of the plane (rectangle, section of an annulus, ...).

(V) Use methods/ideas to attack other types of divisor problems, such as the concentration function

$$\Delta(n) = \max_u \tau(n, u, eu).$$