# VII. The linear sieve

Linear sieve means a sieve problem where $(\Omega)$ holds with $\kappa = 1$.

Input to a sieve

$X$, $g(d)$ and bound on sums of $|r_d|$, e.g. $(R(\theta, A))$

same input to sieve — same bound for $S(\mathcal{A}, \mathcal{P}, z)$

The sets

$$\mathcal{A}^+ = \{n \le x : \lambda(n) = -1\}$$
$$\mathcal{A}^- = \{n \le x ; \lambda(n) = +1\}$$

inputs: $\kappa = 1$, $X = \frac{x}{2}$, $g(d) = \frac{1}{d}$
$(R(\theta, A))$ holds $\forall \theta < 1$, $A > 0$

have the same input, so no sieve method can distinguish between them.

These sets are in fact extremal for all $z \le x$.

Define

$$S^{\pm}(x,s) = S(\mathcal{A}^{\pm}, \mathcal{P}, x^{1/s}), \quad \mathcal{P} = \text{set of all primes.}$$

In particular,

$$(7.1) \qquad S^-(x,s) = 1 \qquad (1 \le s \le 2),$$
$$S^+(x,s) = \pi(x) - \pi(x^{1/s}) \qquad (1 \le s \le 3)$$

Also, for $s \ge 1$,

$$S^{\pm}(x,s) = O(1) + \sum_{\substack{p > x^{1/s} \\ b \ge 1}} \left|\left\{m \le \tfrac{x}{p^b} : \lambda(m) = \pm 1,\ P^-(m) > p\right\}\right| \qquad n = mp^b,\ p = P^-(n)$$

$$(7.2) \qquad = O(1) + \sum_{\substack{x^{1/s} < p^b \le x \\ b \ge 1}} S^{\mp}\left(\frac{x}{p^b}, \frac{\log(x/p^b)}{\log p}\right)$$

$$= \sum_{x^{1/s} < p \le x} S^{\mp}\left(\frac{x}{p}, \frac{\log x}{\log p} - 1\right) + O(x^{1-1/s}), \quad \text{since } \sum_{\substack{p > x^{1/s} \\ b \ge 2}} \frac{1}{p^b} = O(x^{-1/s}).$$

This Buchstab-like identity leads to recursive estimates for $S^{\pm}(x,s)$.
Define $f, F : (0,\infty) \to \mathbb{R}$ to be continuous functions satisfying

$$(7.3) \qquad f(s) = 0 \quad (0 < s \le 2), \qquad F(s) = \frac{2e^{\gamma}}{s} \quad (0 < s \le 3)$$
$$(sf(s))' = F(s-1), \quad (sF(s))' = f(s-1) \qquad (s > 2)$$

In particular,

$$f(s) = 2e^{\gamma} \frac{\log(s-1)}{s} \qquad (2 \le s \le 4)$$

Theorem LS1 Let $N \in \mathbb{N}$. Then, uniformly for $1 \le s \le N$,

$$(7.4) \quad S^+(x,s) = \frac{x/2}{e^{\gamma}\log(x^{1/s})}F(s) + O_N\left(\frac{x}{\log^2 x}\right)$$
$$= \frac{x}{2}V(x^{1/s})F(s) + O_N\left(\frac{x}{\log^2 x}\right);$$

$$(7.5) \quad S^-(x,s) = \frac{x/2}{e^{\gamma}\log(x^{1/s})}f(s) + O_N\left(\frac{x}{\log^2 x}\right)$$
$$= \frac{x}{2}V(x^{1/s})f(s) + O_N\left(\frac{x}{\log^2 x}\right).$$

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Proof By (7.1) and (7.3), (7.4) holds for $N=1,2,3$ and (7.5) holds for $N=1,2$. We prove these by induction on $N$, by illustrating the proof of (7.4) (inductive step) when $N>3$. Say $N \ge 4$, $N-1 < s \le N$. By (7.2),

$$S^+(x,s) = \sum_{x^{1/s}<p\le x} S^-\left(\frac{x}{p}, \frac{\log x}{\log p}-1\right) + O(x^{1-1/N})$$

If $p > x^{1/3}$, then $\frac{\log x}{\log p}-1 < 2$, so $S^-\left(\frac{x}{p}, \frac{\log x}{\log p}-1\right) = 1$. Thus

$$S^+(x,s) = \sum_{x^{1/s}<p\le x^{1/3}} S^-\left(\frac{x}{p}, \frac{\log x}{\log p}-1\right) + \frac{x}{\log x} + O_N\left(\frac{x}{\log^2 x}\right).$$

By the induction assumption, for $x^{1/s}<p\le x^{1/3}$,

$$S^-\left(\frac{x}{p}, \frac{\log x}{\log p}-1\right) = \frac{1}{2e^{\gamma}}\frac{x}{p\log p}f\left(\frac{\log x}{\log p}-1\right) + O_{N-1}\left(\frac{x}{p\log^2 x}\right).$$

Summing on $p$ gives

$$\sum_{x^{1/s}<p\le x^{1/3}} \frac{1}{p} = \log(s/3) + O\left(\frac{s}{\log x}\right) \ll_N 1$$

and

$$\sum_{x^{1/s}<p\le x^{1/3}} \frac{1}{p\log p} f\left(\frac{\log x}{\log p}-1\right) = \underbrace{\int_{x^{1/s}}^{x^{1/3}} \frac{f\left(\frac{\log x}{\log t}-1\right)}{t\log^2 t}\,dt}_{\left[=\frac{1}{\log x}\int_3^s f(v-1)\,dv = \frac{1}{\log x}(sF(s)-3F(3))\right]} + O_N\left(\frac{x}{\log^2 x}\right)$$

$$= \frac{sF(s)-3F(3)}{\log x} + O_N\left(\frac{1}{\log^2 x}\right).$$

Therefore,

$$S^+(x,s) = \frac{x}{2e^{\gamma}\log x}\left(sF(s) - 3F(3) + 2e^{\gamma}\right) + O_N\left(\frac{x}{\log^2 x}\right)$$
$$= \frac{x}{2e^{\gamma}\log x^{1/s}}F(s) + O_N\left(\frac{x}{\log^2 x}\right).$$

The proof for $S^-(x,s)$ is similar.

**Theorem LS** Assume (g), (r) and

$$(\Omega_1) \qquad \prod_{y<p\le w} (1-g(p))^{-1} \le \frac{\log w}{\log y}\left(1+\frac{A}{\log y}\right) \qquad (1.5 \le y \le w),\ A = \text{constant}$$

For any $2 \le z \le D$

$$S(\mathcal{A},\mathcal{P},z) \le X\,V(z)\left\{F\left(\frac{\log D}{\log z}\right) + O_A\left(\frac{\log\log D}{(\log D)^{1/4}}\right)\right\} + \sum_{\substack{d\mid P(z)\\ d\le D}} |r_d|,$$

$$S(\mathcal{A},\mathcal{P},z) \ge X\,V(z)\left\{f\left(\frac{\log D}{\log z}\right) + O_A\left(\frac{\log\log D}{(\log D)^{1/4}}\right)\right\} - \sum_{\substack{d\mid P(z)\\ d\le D}} |r_d| \qquad \blacksquare$$

By Theorem LS1, these estimates are best possible in the sense that $F$ cannot be replaced by a function $\tilde{F}$ with $\tilde{F}(s)<F(s)$ for some $s\ge 1$, and likewise $f$ cannot be replaced by any $\tilde{f}$ with $\tilde{f}(s) > f(s)$ for some $s \ge 1$.

**Theorem fF** We have

(1) $F(u) = 1 + O(e^{-u\log u + 2u})$, $f(u) = 1 + O(e^{-u\log u+2u})$ $\qquad (u>0)$

(2) $0 \le f(u) < 1 < F(u)$ $\quad (u>0)$, $F(u)\downarrow 1$, $f(u)\uparrow 1$.

(3) $$0 \le \left\{\begin{matrix} F(u_1)-F(u_2) \\ f(u_2)-f(u_1)\end{matrix}\right\} \le 2e^{\gamma}\,\frac{u_2-u_1}{u_1} \qquad (1\le u_1 \le u_2).$$

Proof: Introduce $P(u) = F(u)+f(u)$, $Q(u) = u(F(u)-f(u))$. By (7.3), we have

(i) $$uP(u) = Q(u) = 2e^{-\gamma} \qquad (0<u\le 2),$$

(ii) $$uP'(u) = P(u-1)-P(u) \qquad (u\ge 2),$$

(iii) $$((u-1)Q(u))' = Q(u)-Q(u-1) \qquad (u\ge 2).$$

From (ii),

$$|P'(u)| = \left|\frac{1}{u}\int_{u-1}^{u} P'(t)\,dt\right| \le \frac{1}{u}\max_{u-1\le t\le u}|P'(t)|.$$

Integrating (iii) from $u=2$ to $v$ and using $Q(2) = 2e^{\gamma} = \int_1^2 Q(v)\,dv$ (from (i)) gives

$$|(v-1)Q(v)| = \left|\int_{v-1}^{v} Q(t)\,dt\right| \le \max_{v-1\le t\le v}|Q(t)|.$$

Thus, for $\phi\in\{P', Q\}$, we have $|\phi(u)| \le \frac{1}{u-1}\max_{u-1\le t\le u}|\phi(t)|$. Letting $M_\phi(u) = \max_{t\ge u}|\phi(t)|$, we have $M_\phi(u) \le \frac{M_\phi(u-1)}{u-1}$, hence by iteration,

$$M_\phi(u) \le \frac{M(u-\lfloor u\rfloor+1)}{(u-1)(u-2)\cdots(u-\lfloor u\rfloor+1)} \ll \frac{1}{\Gamma(u)} \asymp \frac{u^{1/2}}{e^{u\log u - u}}.$$

Hence $Q(u) \ll \frac{1}{\Gamma(u)}$ and $P'(u) \ll \frac{1}{\Gamma(u)}$. Integrating the second inequality gives

$$P(u) = c + O\left(\int_u^\infty \frac{dv}{\Gamma(v)}\right) = c + O\left(\frac{1}{\Gamma(u)}\right), \text{ some constant } c = \lim_{u\to\infty} P(u).$$

Then

$$f(u), F(u) = \frac{c}{2} + O\left(\frac{1}{\Gamma(u)}\right) = \frac{c}{2} + O\left(e^{-u\log u + 2u}\right).$$

Finally, we determine $c$. By Theorem LS1, for any fixed $s$,

$$S^+(x,s) + S^-(x,s) \sim \frac{x}{2}(f(s)+F(s))V(x^{1/s}) \sim \frac{x}{2}P(s)\prod_{p\le x^{1/s}}\left(1-\frac{1}{p}\right).$$

But also

$$S^+(x,s) + S^-(x,s) = \Phi(x, x^{1/s}) = x\prod_{p\le x^{1/s}}\left(1-\frac{1}{p}\right)\left(1+O\left(\frac{1}{\log x} + e^{-s}\right)\right).$$

by the Corollary of Theorem BH.4. Letting $s\to\infty$ shows $c=2$ and proves (1).

From $(v-1)Q(v) = \int_{v-1}^{v} Q(t)\,dt$, we see that $Q(u) > 0$, so $F(u) > f(u)$ for all $u>0$.

Since

$$uF'(u) = f(u-1) - F(u) < f(u-1) - f(u),$$

$F'$ is continuous. So, if $F$ is not strictly decreasing, let $u_0 = \min\{u : F'(u) = 0\}$. Then

$$0 < f(u_0-1) - f(u_0) = -\int_{u_0-1}^{u_0} f'(t)\,dt.$$

So there is some $u_1 \in (u_0-1, u_0)$ with $f'(u_1) < 0$. Then

$$0 > u_1 f'(u_1) = F(u_1-1) - f(u_1) \ge F(u_1-1) - F(u_1) = -F'(u_2)$$

for some $u_2 \in (u_1-1, u_1)$. This contradiction shows that $F \downarrow 1$. Now

$$uf'(u) = F(u-1) - f(u) > F(u-1) - F(u) > 0 \quad (u>2).$$

and (2) follows. To prove (3), the Mean Value Theorem gives

$$F(u_1) - F(u_2) = (u_2-u_1)(-F'(u_3)) \text{ for some } u_3 \in [u_1, u_2]$$

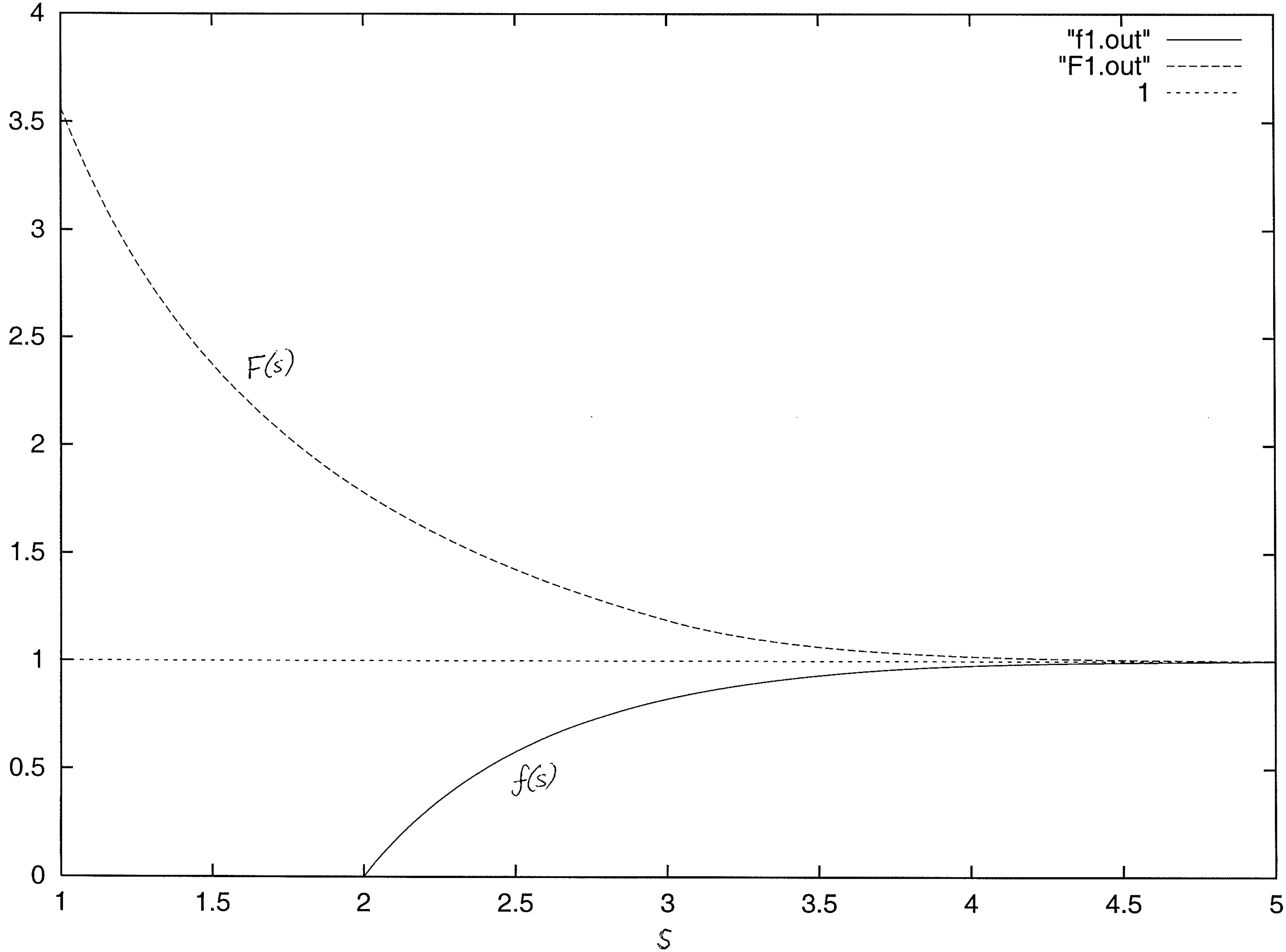
If $u_3 \le 2$, then $F'(u_3) = \frac{-2e^\gamma}{u_3^2} \ge \frac{-2e^\gamma}{u_3} \ge -\frac{2e^\gamma}{u_1}$. If $u_3 > 2$, then

$$-F'(u_3) = \frac{F(u_3) - f(u_3-1)}{u_3} \le \frac{F(u_3)}{u_3} \le \frac{e^\gamma}{u_1}.$$

For the $f$ inequality, if $u_2 \le 2$ then $f(u_2) - f(u_1) = 0$. Otherwise, $f(u_2) - f(u_1) = (u_2-u_1)f'(u_3)$ for some $u_3 \in [u_1, u_2]$, and

$$f'(u_3) = \frac{F(u_3-1) - f(u_3)}{u_3} \le \frac{F(1)}{u_3} \le \frac{2e^\gamma}{u_1}.$$

This proves (3).∎



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## Application ; Twin almost primes

**Theorem** There are infinitely many primes $p$ (actually $\gg \frac{x}{\log^2 x}$ such $p \le x$) with

(a) $\Omega(p+2) \le 4$ (b) $\Omega(p+2) \le 3$

**Proof** Let $\mathcal{A} = \{p+2 : p \le x\}$, $\mathcal{P}$ = all primes $> 2$, $X = li(x)$. Then

$$|\mathcal{A}_d| = \pi(x; d, -2) = \frac{li(x)}{\phi(d)} + r_d \quad \text{for } (d,2)=1,$$

$g(d) = \frac{1}{\phi(d)}$. We have for $y \ge 2$

$$\frac{V(y)}{V(w)} = \prod_{y<p\le w} (1-g(p))^{-1} = \prod_{y<p\le w} \left(\frac{p-1}{p-2}\right)$$

$$= \prod_{y<p\le w} \left(1-\frac{1}{p}\right)^{-1} \cdot \left(1+\frac{1}{p(p-2)}\right) = \frac{\log w}{\log y}\left(1+O\left(\frac{1}{\log y}\right)\right)$$

by Mertens' estimate. So $(\Omega_1)$ holds with some absolute $A$.

By Theorem BV, if $B$ is large enough and $D = x^{\frac{1}{2}}(\log x)^{-B}$, then

$$\sum_{\substack{d \le D \\ 2\nmid d}} |r_d| \ll \frac{x}{(\log x)^4}.$$

Fix $0 < \delta \le \frac{1}{10}$, $u = 4(1+\delta)$ and $z = x^{\frac{1}{u}}$. Then

$$s := \frac{\log D}{\log z} = 4(1+\delta)\left\{\frac{1}{2} - B\frac{\log\log x}{\log x}\right\} \ge 2 + \tfrac{3}{2}\delta \quad (x \ge x_0(\delta))$$

and

$$f(s) = 2e^{\gamma}\frac{\log(s-1)}{s} \ge 2e^{\gamma}\frac{\log(1+\frac{3}{2}\delta)}{2+\frac{3}{2}\delta} \ge 2.3\delta$$

By Theorem LS,

$$\#\{p \le x : \Omega(p+2) \le 4\} \ge S(\mathcal{A}, \mathcal{P}, z)$$

$$\ge li(x)V(z)\left\{f(s) + O\left(\frac{1}{(\log x)^{1/5}}\right)\right\} - O\left(\frac{x}{(\log x)^4}\right).$$

Since

$$V(z) = \prod_{2<p\le z}\frac{p-2}{p-1} = \prod_{p\le z}\left(1-\frac{1}{p}\right)\cdot 2\prod_{2<p\le z}\left(1-\frac{1}{(p-1)^2}\right)$$

$$\sim C\prod_{p\le z}\left(1-\frac{1}{p}\right) \sim \frac{Ce^{-\gamma}}{\log z} \quad (z\to\infty),$$ $C$ = twin prime const. $= 1.3203\ldots$

$$\#\{p \le x : \Omega(p+2) \le 4\} \ge \#\{p \le x : P^-(p+2) > x^{\frac{1}{4(1+\delta)}}\}$$

$$\ge 6\delta\frac{x}{\log^2 x} \quad (x \ge x_1(\delta)).$$

This proves part (a).

We cannot deduce (b) by taking $\delta=0$, but we can take $\delta$ very small and use an argument of R.C. Vaughan:

If $n$ is counted by $S(\mathcal{A},\mathcal{P},z)$, and if $\Omega(n)=4$, then $n=p_1p_2p_3p_4$ with $p_1\geq p_2\geq p_3\geq p_4\geq z=x^{1/u}=x^{\frac{1}{4(1+\delta)}}$. Also,

$$p_1=\frac{n}{p_2p_3p_4}\leq\frac{x+2}{x^{3/u}}\leq x^{\frac{1+4\delta}{4+4\delta}}+1<x^{\frac{1}{4}+\delta}.$$

So all $p_i$ are very close to $x^{1/4}$. The number of such $n$ is

$$\leq\sum_{x^{\frac{1}{4+4\delta}}\leq p_4\leq p_3\leq p_2\leq x^{\frac{1}{4}+\delta}}\#\left\{p_1\leq\frac{x+2}{p_2p_3p_4}: p_1(p_2p_3p_4)-2 \text{ is prime}\right\},$$ use homework problem 1

$$\ll\sum_{p_2,p_3,p_4}\frac{x/p_2p_3p_4}{\log^2\left(\frac{x}{p_2p_3p_4}\right)}\frac{2p_2p_3p_4}{\phi(2p_2p_3p_4)}\quad ;\quad p_2p_3p_4\leq x^{\frac{3}{4}+3\delta}$$

$$\ll\frac{x}{\log^2 x}\cdot\sum_{x^{\frac{1}{4+4\delta}}<p_4,p_3,p_2\leq x^{\frac{1}{4}+\delta}}\frac{1}{p_2p_3p_4}$$

$$\ll\frac{x}{\log^2 x}\left(\log\left(\frac{\frac{1}{4}+\delta}{\frac{1}{4+4\delta}}\right)+O\left(\frac{1}{\log x}\right)\right)^3\ll\frac{x\delta^3}{\log^2 x}\quad (x\geq x_2(\delta)).$$

Hence,

$$\#\{p\leq x:\Omega(p+2)\leq 3\}\geq\frac{x}{\log^2 x}\left(6\delta-O(\delta^3)\right)$$
$$\geq 5\delta\frac{x}{\log^2 x}$$

for sufficiently small $\delta$ and $x$ sufficiently large. This proves (b).

$$\left.\begin{array}{l}\underline{\text{Theorem}}\ \text{(a) } \Omega(n^2+1)\leq 4 \text{ for infinitely many } n\in\mathbb{N}\\ \text{(b) } \Omega(n^2+1)\leq 3 \text{ for infinitely many } n\in\mathbb{N}\end{array}\right\}\text{ homework}$$

4-11-11. #30



# Rosser - Iwaniec sieve

Lemma (Buchstab's identity)

For $z' \le z$, $S(\mathcal{A},\mathcal{P},z) = S(\mathcal{A},\mathcal{P},z') - \sum_{p \mid P(z)/P(z_1)} S(\mathcal{A}_p,\mathcal{P},p-1)$

Proof If $(n,P(z'))=1$, then either $(n,P(z))=1$ or $(n,P(z))$ has a prime factor in $(z',z]$. Let $p$ be the smallest such prime factor.

In particular,

(7.6) $$S(\mathcal{A},\mathcal{P},z) = |\mathcal{A}| - \sum_{p \mid P(z)} S(\mathcal{A}_p,\mathcal{P},p-1), \quad \mathcal{A}_p = \{n \in \mathcal{A} : p \mid n\}.$$

Iterating (Bu1) gives

$$S(\mathcal{A},\mathcal{P},z) = |\mathcal{A}| - \sum_{p_1 \mid P(z)} \Big( |\mathcal{A}_{p_1}| - \sum_{\substack{p_2 \mid P(z) \\ p_2 < p_1}} S(\mathcal{A}_{p_1 p_2},\mathcal{P},p_2-1) \Big).$$

For any $y_2 = y_2(p_1) \le p_1$, we have the lower bound

$$S(\mathcal{A},\mathcal{P},z) \ge |\mathcal{A}| - \sum_{p_1 \mid P(z)} \Big\{ |\mathcal{A}_{p_1}| - \sum_{\substack{p_2 \mid P(z) \\ p_2 < y_2}} S(\mathcal{A}_{p_1 p_2},\mathcal{P},p_2-1) \Big\}$$

$$= |\mathcal{A}| - \sum_{p_1 \mid P(z)} \Big\{ |\mathcal{A}_{p_1}| - \sum_{\substack{p_2 \mid P(z) \\ p_2 < y_2}} \Big\{ |\mathcal{A}_{p_1 p_2}| - \sum_{\substack{p_3 \mid P(z) \\ p_3 < p_2}} \Big\{ |\mathcal{A}_{p_1 p_2 p_3}| - \sum_{\substack{p_4 \mid P(z) \\ p_4 < y_4}} S(\mathcal{A}_{p_1 \cdots p_4},\mathcal{P},p_4-1) \Big\}\Big\}\Big\}$$

for any $y_4 = y_4(p_1,p_2,p_3) \le p_4$. Continuing in this way yields a general lower bound sieve (by choosing $y_2, y_4, y_6, \cdots$): let

$$\mathcal{D}^- = \{ p_1 \cdots p_\ell : p_1 > p_2 > \cdots > p_\ell \ (\ell \text{ variable}),\ p_m < y_m \ (m \text{ even}) \}.$$

Then $\lambda^-(d) = \mu(d) 1_{\mathcal{D}^-}(d)$ is a lower bound sieve. Similarly, if $y_1, y_3 = y_3(p_1,p_2) \le p_2$, etc, and

$$\mathcal{D}^+ = \{ p_1 \cdots p_\ell : p_1 > p_2 > \cdots > p_\ell,\ p_m < y_m \ (m \text{ odd}) \},$$

then $\lambda^+(d) = \mu(d) 1_{\mathcal{D}^+}(d)$ is an upper bound sieve. That is,

$$\sum_{\substack{d \mid P(z) \\ d \in \mathcal{D}^-}} \mu(d) |\mathcal{A}_d| \le S(\mathcal{A},\mathcal{P},z) \le \sum_{\substack{d \mid P(z) \\ d \in \mathcal{D}^+}} \mu(d) |\mathcal{A}_d|.$$

Similarly, truncating the iteration process at primes $p_i > z'$, where $z' \le z$, yields

(7.7) $$\sum_{\substack{d \mid P(z)/P(z') \\ d \in \mathcal{D}^-}} \mu(d) S(\mathcal{A}_d,\mathcal{P},z') \le S(\mathcal{A},\mathcal{P},z) \le \sum_{\substack{d \mid P(z)/P(z') \\ d \in \mathcal{D}^+}} \mu(d) S(\mathcal{A}_d,\mathcal{P},z')$$

## The Rosser–Iwaniec choice for $y_m$

Let $\beta \geq 1$ be fixed, $Y$ a parameter, and

$$(y_m) \qquad y_m = \left(\frac{Y}{p_1 \cdots p_{m-1}}\right)^{\frac{1}{\beta+1}}$$

This gives a sieve called the "beta sieve". For each dimension $\kappa > 0$, there is an optimum choice of $\beta = \beta_\kappa$ (the theory is worked out completely in papers by Diamond, Halberstam & Richert; see also "A higher dimensional sieve method" by Diamond & Halberstam, Cambridge Tracts in Mathematics #177, 2008). For $\kappa = 1$, we take $\beta = 2$.

<u>Proposition</u> If $\beta \geq 1$ and $(y_m)$, then $d \in \mathcal{D}^\pm \Rightarrow d < Y$.

<u>Proof</u> Let $d = p_1 \cdots p_\ell$, $p_1 > \cdots > p_\ell$. If $\ell$ is even, then

$$d \in \mathcal{D}^- \Rightarrow d < p_1 \cdots p_{\ell-1} p_\ell^{\beta+1} < Y,$$
$$d \in \mathcal{D}^+ \Rightarrow d < p_1 \cdots p_{\ell-1}^{\beta+1} < Y,$$

and similarly if $\ell$ is odd.

For general $\kappa$, $\beta$ is called the "sifting limit", since a nontrivial lower bound for $S(\mathcal{A}, \mathcal{P}, z)$ requires $z^\beta \leq Y$. The sifting limit for $\kappa = 1$ is $\beta = 2$ (recall $f(s) = 0$ for $s \leq 2$) and $\beta = 1$ for $\kappa \leq \frac{1}{2}$ (the best we can hope for).

4-13-11, #31

## Idea of proof of Theorem LS

Start with (7.7) with a "small" $z'$, use the Fundamental Lemma on each summand. Specifically, take

$$(z') \qquad z' = \exp\left\{\left(\frac{\log^3 D}{\log\log D}\right)^{1/4}\right\}.$$

If $z \leq z'$, the Fundamental Lemma (Theorem BH.4) gives

$$S(\mathcal{A}, \mathcal{P}, z) = X V(z)\left(1 + O\left(s^{-\frac{1}{3}s}\right)\right) + R, \quad |R| \leq \sum_{\substack{d \leq D \\ d \mid P(z)}} |r_d|,$$

where $s = \frac{\log D}{\log z} \geq (\log D \log\log D)^{1/4}$. Since $|F(s)-1|, |f(s)-1| \ll \frac{1}{\log D}$, Theorem LS follows in this case.

Now assume $z > z'$. We'll concentrate on proving the lower estimate in theorem **LS**; the upper estimate has a similar proof. In (7.7), use the Fundamental Lemma to estimate each $S(\mathcal{A}_d, \mathcal{P}, z')$. In the set $\mathcal{D}^-$, we take

$$\text{(Y)} \quad s' = \log\log D, \quad Y = D(z')^{-s'} = D\exp\left\{-(\log D \log\log D)^{3/4}\right\}.$$

In the application of Theorem BH.4,

$$S(\mathcal{A}_d, \mathcal{P}, z') = Xg(d)V(z')\left\{1 + O\left((s')^{-\frac{1}{3}s'}\right)\right\} - \sum_{\substack{m \le (z')^{s'} \\ m \mid P(z')}} |r_{dm}|,$$

so by (7.7),

$$S(\mathcal{A}, \mathcal{P}, z) \ge XV(z') \sum_{\substack{d \mid P(z)/P(z') \\ d \in \mathcal{D}^-}} \mu(d)g(d)\left\{1 + O\left(\frac{1}{\log^2 D}\right)\right\} - \sum_{\substack{d \le Y \\ d \mid P(z)/P(z')}} \sum_{\substack{m \le (z')^{s'} \\ m \mid P(z')}} |r_{dm}|.$$

We have $dm \le Y(z')^{s'} = D$. Also, by $(\Omega_1)$,

$$\sum_{\substack{d \mid P(z)/P(z') \\ d \in \mathcal{D}^-}} g(d) \le \prod_{z' < p \le z} (1 + g(p)) \le \prod_{z' < p \le z} (1 - g(p))^{-1}$$
$$= \frac{V(z')}{V(z)} = \frac{V(z)}{V(z')}\left(\frac{V(z')}{V(z)}\right)^2 \ll \frac{V(z)}{V(z')}\left(\frac{\log z}{\log z'}\right)^2 \ll \frac{V(z)}{V(z')}\left(\frac{\log\log D}{\log D}\right)^{\frac{1}{2}}.$$

Therefore,

$$(7.8) \quad S(\mathcal{A}, \mathcal{P}, z) \ge XV(z') \sum_{\substack{d \mid P(z)/P(z') \\ d \in \mathcal{D}^-}} \mu(d)g(d) + O\left(XV(z)(\log D)^{-1}\right) - \sum_{\substack{d \mid P(z) \\ d \le D}} |r_d|.$$

It remains to bound the first sum in (7.8).

**Lemma LS2** Suppose $(\Omega_1)$, $2 \le u \le w$ and $B(t)$ is non-negative, continuous and increasing on $[u, w]$. Then

$$\frac{1}{\log w} \sum_{u < p \le w} g(p)\frac{V(p)}{V(w)}B(p) \le \int_u^w \frac{B(t)}{t\log^2 t} + A\frac{B(w)}{\log^2 u}.$$

**Proof** First,

$$V(w) = \sum_{d \mid P(w)} \mu(d)g(d) = 1 + \sum_{p \le w} \sum_{\substack{d \mid P(w) \\ P^+(d) = p}} \mu(d)g(d)$$
$$= 1 - \sum_{p \le w} g(p) \sum_{t \mid P(p)} \mu(t)g(t) = 1 - \sum_{p \le w} g(p)V(p).$$

Hence

$$(7.9) \quad V(u) - V(w) = \sum_{u < p \le w} g(p)V(p).$$

Then

$$\sum_{u<p\le W} g(p)\frac{V(p)}{V(w)}B(p) = \sum_{u<p\le w} g(p)\frac{V(p)}{V(w)}\left(B(u)+\int_u^p dB(t)\right)$$
$$= B(u)\left(\frac{V(u)}{V(w)}-1\right)+\int_u^w\left(\sum_{t<p\le w} g(p)\frac{V(p)}{V(w)}\right)dB(t)$$
$$= B(u)\left(\frac{V(u)}{V(w)}-1\right)+\int_u^w\left(\frac{V(t)}{V(w)}-1\right)dB(t).$$

By $(\Omega_1)$, $\frac{V(t)}{V(w)} \le \frac{\log w}{\log t}\left(1+\frac{A}{\log t}\right)$. Hence

$$\frac{1}{\log w}\sum_{u<p\le w} g(p)\frac{V(p)}{V(w)}B(p) \le B(u)\left(\frac{1}{\log u}+\frac{A}{\log^2 u}-\frac{1}{\log w}\right)+\int_u^w\left(\frac{1}{\log t}+\frac{A}{\log^2 t}-\frac{1}{\log w}\right)dB(t)$$
$$= \frac{AB(w)}{\log^2 w}+\int_u^w B(t)\left(\frac{1}{t\log^2 t}+\frac{2A}{t\log^3 t}\right)dt$$
$$\le \frac{AB(w)}{\log^2 w}+\int_u^w \frac{B(t)}{t\log^2 t}dt + AB(w)\left(\frac{1}{\log^2 u}-\frac{1}{\log^2 w}\right). \blacksquare$$

<u>Corollary LS3</u>

(a) $\displaystyle\sum_{u<p\le w} g(p)V(p)F\left(\frac{\log Y}{\log p}-1\right)+V(w)f\left(\frac{\log Y}{\log w}\right)-V(u)f\left(\frac{\log Y}{\log u}\right) \le 2e^{\gamma}A\left(\frac{\log w}{\log^2 u}\right)V(w)$ $\quad(2\le u\le w\le Y^{\frac{1}{2}})$

(b) $\displaystyle V(u)F\left(\frac{\log Y}{\log u}\right)-V(w)F\left(\frac{\log Y}{\log w}\right)-\sum_{u<p\le w} g(p)V(p)f\left(\frac{\log Y}{\log p}-1\right) \le 2e^{\gamma}A\left(\frac{\log w}{\log^2 u}\right)V(w)$ $\quad(2\le u\le w\le Y)$

<u>Proof</u> Use Lemma LS2 with (a) $B(t)=F\left(\frac{\log Y}{\log t}-1\right)-1$, (b) $B(t)=1-f\left(\frac{\log Y}{\log t}-1\right)$. We get, by (7.9),

$$\sum_{u<p\le w} g(p)V(p)F\left(\frac{\log Y}{\log p}-1\right) \le V(u)-V(w)+V(w)\log w\left(\frac{A}{\log^2 u}\left[F\left(\frac{\log Y}{\log w}-1\right)-1\right]+\frac{1}{\log w}-\frac{1}{\log u}+\int_u^w\frac{F\left(\frac{\log Y}{\log t}-1\right)}{t\log^2 t}dt\right)$$
$$= V(u)+V(w)\log w\left(\frac{A}{\log^2 u}\left[F\left(\frac{\log Y}{\log w}-1\right)-1\right]-\frac{1}{\log u}+\frac{1}{\log Y}\int_{\frac{\log Y}{\log w}}^{\frac{\log Y}{\log u}} F(s-1)\,ds\right)$$
$$\le V(u)+V(w)\log w\left(\frac{A(2e^{\gamma}-1)}{\log^2 u}-\frac{1}{\log u}+\frac{1}{\log u}f\left(\frac{\log Y}{\log u}\right)-\frac{1}{\log w}f\left(\frac{\log Y}{\log w}\right)\right)$$
$$\le -V(w)f\left(\frac{\log Y}{\log w}\right)+V(u)f\left(\frac{\log Y}{\log u}\right)+\frac{A(2e^{\gamma}-1)V(w)\log w}{\log^2 u}+\left(1-f\left(\frac{\log Y}{\log u}\right)\right)V(w)\left(\frac{V(u)}{V(w)}-\frac{\log w}{\log u}\right).$$

By $(\Omega_1)$, $\frac{V(u)}{V(w)} \le \frac{\log w}{\log u}+\frac{A\log w}{\log^2 u}$, and also $1-f\left(\frac{\log Y}{\log u}\right)\le 1$. This proves (a).

The proof of (b) is similar.

Lemma LS4 Let $P = P(z)/P(z')$. Then

(i) $$\sum_{\substack{d|P \\ d\in\mathcal{D}^-}} \mu(d)g(d) \geq \frac{V(z)}{V(z')}\left\{ f\left(\frac{\log Y}{\log z}\right) - 2e^{\gamma}A\left(1+\frac{A}{\log^2 z'}\right)\frac{(\log z)^2}{(\log z')^3}\right\} \qquad (2\le z'\le z\le Y^{\frac{1}{2}})$$

(ii) $$\sum_{\substack{d|P \\ d\in\mathcal{D}^+}} \mu(d)g(d) \leq \frac{V(z)}{V(z')}\left\{ F\left(\frac{\log Y}{\log z}\right) + 2e^{\gamma}A\left(1+\frac{A}{\log^2 z'}\right)\frac{(\log z)^2}{(\log z')^3}\right\} \qquad (2\le z'\le z\le Y).$$

Proof of (i)

Let $\phi^r = F$ ($r$ even), $\phi^r = f$ ($r$ odd). Since $f(u)<1<F(u)$ for all $u$,

$$\sum_{\substack{d|P \\ d\in\mathcal{D}^-}} \mu(d)g(d) \geq \sum_{\substack{d|P \\ d\in\mathcal{D}^-}} \mu(d)g(d)\,\phi^{\omega(d)+1}\left(\frac{\log(Y/d)}{\log z'}\right) =: T.$$

For $z\le Y^{1/2}$ we have

(*) $$V(z)f\left(\frac{\log Y}{\log z}\right) - V(z')T = E_1^- + \sum_{r=1}^{\infty}(E_{r+1}^- - E_r^-),$$

where $$E_r^- = V(z)f\left(\frac{\log Y}{\log z}\right) - V(z')\sum_{\substack{d|P \\ d\in\mathcal{D}^-,\,\omega(d)<r}} \mu(d)g(d)\,\phi^{\omega(d)+1}\left(\frac{\log(Y/d)}{\log z'}\right)$$
$$- \sum_{\substack{d|P \\ d\in\mathcal{D}^-,\,\omega(d)=r}} \mu(d)g(d)V(P^-(d))\,\phi^{\omega(d)+1}\left(\frac{\log(Y/d)}{\log P^-(d)}\right),$$

since clearly $E_r^-$ is constant for large $r$, and $\lim_{r\to\infty} E_r^- = V(z)f\left(\frac{\log Y}{\log z}\right) - V(z_1)T$.

By Corollary LS3 (a) with $u=z'$, $w=z$,

$$E_1^- = V(z)f\left(\frac{\log Y}{\log z}\right) - V(z')f\left(\frac{\log Y}{\log z'}\right) + \sum_{z'<p\le z} g(p)V(p)F\left(\frac{\log Y}{\log p}-1\right)$$
$$\le 2e^{\gamma}A\left(\frac{\log z}{\log^2 z'}\right)V(z).$$

Next,

$$E_{r+1}^- - E_r^- = -V(z')\sum_{\substack{d|P,\,d\in\mathcal{D}^- \\ \omega(d)=r}} (-1)^r g(d)\,\phi^{r+1}\left(\frac{\log(Y/d)}{\log z'}\right)$$

(!) $$+ (-1)^r \sum_{\substack{d|P,\,d\in\mathcal{D}^- \\ \omega(d)=r}} g(d)V(P^-(d))\,\phi^{r+1}\left(\frac{\log(Y/d)}{\log P^-(d)}\right)$$

$$+ (-1)^r \sum_{\substack{d'|P,\,d'\in\mathcal{D}^- \\ \omega(d')=r+1}} g(d')V(P^-(d'))\,\phi^{r+2}\left(\frac{\log(Y/d')}{\log P^-(d')}\right).$$

In the third sum,
write $d' = dp$, where $d \in \mathcal{D}^-$, $p = P^-(d')$. If $r$ is odd,

$$f\left(\frac{\log(Y/d')}{\log p}\right) > 0 \iff \frac{\log(Y/d')}{\log p} > 2 \iff dp^3 < Y \iff dp \in \mathcal{D}^-.$$

Thus, the third sum is always

$$\leq (-1)^r \sum_{\substack{d \mid P,\, d \in \mathcal{D}^- \\ \omega(d) = r}} g(d) \sum_{z' \leq p < P^-(d)} V(p) g(p) \phi^r\left(\frac{\log(Y/d)}{\log p} - 1\right),$$

and hence

$$E^-_{r+1} - E^-_r \leq (-1)^{r+1} \sum_{\substack{d \mid P,\, d \in \mathcal{D}^- \\ \omega(d) = r}} g(d) \Big[ V(z') \phi^{r+1}\left(\frac{\log(Y/d)}{\log z'}\right) - V(P^-(d)) \phi^{r+1}\left(\frac{\log(Y/d)}{\log P^-(d)}\right) - \sum_{z' < p < P^-(d)} g(p) V(p) \phi^r\left(\frac{\log(Y/d)}{\log p} - 1\right) \Big].$$

By Corollary LS 3 (with $u = z'$, $w = P^-(d)$),

$$E^-_{r+1} - E^-_r \leq (-1)^{r+1} \sum_{\substack{d \mid P,\, d \in \mathcal{D}^- \\ \omega(d) = r}} g(d) \left[ 2Ae^\gamma \frac{\log P^-(d)}{\log^2 z'} V(P^-(d)) \right].$$

Inserting this bound into $(*)$ yields

$$(\cdot\cdot) \quad E^-_1 + \sum_{r=1}^{\infty} (E^-_{r+1} - E^-_r) \leq \frac{2Ae^\gamma}{(\log z')^2} V(z) \log z \left[ 1 + \sum_{r=1}^{\infty} \sum_{\substack{d \mid P,\, d \in \mathcal{D}^- \\ \omega(d) = r}} g(d) \frac{V(P^-(d)) \log P^-(d)}{V(z) \log z} \right].$$

Since $\displaystyle \frac{V(P^-(d))}{V(z)} \leq \frac{\log z}{\log P^-(d)} \left(1 + \frac{A}{\log P^-(d)}\right) \leq \frac{\log z}{\log P^-(d)} \left(1 + \frac{A}{\log z'}\right)$ by $(\Omega_1)$,

the bracketed [ ] expression in $(\cdot\cdot)$ is $\displaystyle \leq \left(1 + \frac{A}{\log z'}\right) \sum_{d \mid P} g(d) \leq \left(1 + \frac{A}{\log z'}\right) \prod_{z' \leq p \leq z} \frac{1}{1 - g(p)}$

$$\leq \frac{\log z}{\log z'} \left(1 + \frac{A}{\log z'}\right)^2.$$

Inserting this into $(\cdot\cdot)$ gives

$$E^-_1 + \sum_{r=1}^{\infty} (E^-_{r+1} - E^-_r) \leq 2e^\gamma A \left(1 + \frac{A}{\log z'}\right)^2 V(z) \frac{(\log z)^2}{(\log z')^3}.$$

Part (i) now follows from $(*)$.

Endgame. Proof of Theorem LS (lower bound)

By (7.8) and Lemma LS4(i)

$$S(A,\mathcal{P},z) \geq X\,V(z)\left\{f\left(\frac{\log Y}{\log z}\right) + O\left(\frac{(\log z)^2}{(\log z')^3}\right) + O\left(\left(\frac{\log\log D}{\log D}\right)^{1/2}\right)\right\} - \sum_{\substack{d \leq D \\ d \mid P(z)}} |r_d|$$

Since $z \leq D$, by $(z')$,

$$\frac{(\log z)^2}{(\log z')^3} \leq \frac{(\log\log D)^{3/4}}{(\log D)^{1/4}}.$$

Also, by Theorem fF(3) and (Y),

$$0 \leq f\left(\frac{\log D}{\log z}\right) - f\left(\frac{\log Y}{\log z}\right) \leq 2e^{\gamma}\,\frac{\log(D/Y)}{\log D} = 2e^{\gamma}\,\frac{(\log\log D)^{3/4}}{(\log D)^{1/4}}.$$

Therefore,

$$S(A,\mathcal{P},z) \geq X\,V(z)\left\{f\left(\frac{\log D}{\log z}\right) + O\left(\frac{(\log\log D)^{3/4}}{(\log D)^{1/4}}\right)\right\} - \sum_{\substack{d \mid P(z) \\ d \leq D}} |r_d| \blacksquare$$

The upper bound proof is similar, using Lemma LS4(ii).

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Application. Almost primes in short intervals.

Theorem For any $k \geq 1$, $\varepsilon > 0$, if $x$ is sufficiently large then

$$\#\{x \leq n \leq x + x^{\frac{2}{k+1}+\varepsilon} : \Omega(n) \leq k\} \gg_{\varepsilon,k} x^{\frac{2}{k+1}+\varepsilon} / \log x.$$