Continuity

1-D $f: \mathbb{R}\to\mathbb{R}$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

2-D $f: \mathbb{R}^2\to\mathbb{R}$ is continuous at $(a,b)$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Three requirements:

1. $f$ is defined at $(a,b)$
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3. the two quantities in (1) and (2) are equal

Examples

A) $f(x,y) = \frac{\sin(xy)}{xy}$ discontinuous on $x$-axis, $y$-axis: fails (1)

B) $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$ satisfies (1) everywhere but fails (2) on $x,y$-axes:

- e.g. at $(0,0)$, approach along pos. $x$-axis: $\lim_{x \to 0^+} = 0$
- approach along $y=x(x>0)$: $\lim_{x \to 0^+} \frac{\sin(x^2)}{x^2} = 1 \Rightarrow$ limit DNE

C) $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & xy \neq 0 \\ 1 & xy = 0 \end{cases}$

This is continuous for all $x,y$.

Reason: when $ab=0$, $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{u\to 0} \frac{\sin u}{u} = 1 = f(a,b)$ (for path off-axes)
Partial Derivatives, Intro

\[ f(x,y) = x^2 + y^2 \text{ at point } (0,1) \]

1) \( \frac{\partial f}{\partial x} \) or \( f_x \). Fix \( y \), derivative with respect to \( x \) = 2x

2) \( \frac{\partial f}{\partial y} \) or \( f_y \). Fix \( x \), derivative with respect to \( y \) = 2y

Example. \( f(x,y) = x^2 y + \sin x \). \( f_x = 2xy + \cos x \). \( f_y = x^2 \)

\[ f(x,y,z) = e^{xy} \sin z \]. \( f_x = ye^{xy} \sin z \), \( f_y = xe^{xy} \sin z \), \( f_z = e^{xy} \cos z \)

Higher derivatives

\( f_{xx} \) or \( \frac{\partial^2 f}{\partial x^2} \) means \( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \) or \( (f_x)_x \)

Meaning of \( f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} \)

Example. \( f(x,y) = \tan^{-1}(x^2y) \). Find \( f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx} \)

Always \( f_{xy} = f_{yx} \) (Clairaut's Theorem) if \( f_{xy} \) and \( f_{yx} \) are continuous

Preview

\( f_x \) and \( f_y \) together help us find the tangent plane to a surface.