Multidimensional Calculus

LIMITS

Derivatives
slopes (directional)
rate of change

Integrals
Double Triple Line

Single-variable calculus review

derivative of \( f \) at \( x \) : \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), type \( \frac{0}{0} \) limit

\( \lim_{x \to a} g(x) = L \) means "for all \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that whenever \( 0 < |x-a| < \delta \), \( |g(x) - L| < \varepsilon \)."

Rough meaning: As \( x \) gets closer to \( a \), \( g(x) \) gets closer to \( L \)

Example: \( g(x) = \frac{1}{x} \) \( \lim_{x \to 0} g(x) \) DNE

Limits different as you approach zero from the left and the right.

Two-dimensional limits (much more complicated!)

\( \lim_{(x,y) \to (a,b)} g(x,y) = L \) means "for all \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that if \( 0 < |(x,y)-(a,b)| < \delta \) then \( |g(x,y) - L| < \varepsilon \)."

Geometrically: \( \delta \) inside the circle, \( (a,b) \)
This means: As \((x,y)\) gets closer to \((a,b)\), no matter how \(g(x,y)\) gets closer to \(L\) on any path/curve/approach.

\[
\text{Example:} \quad g(x,y) = \frac{xy^3}{x^4+y^4}
\]

**Approach:**
- **x-axis:** \(x=0, y \to 0\):
  \[
  \lim_{y \to 0} g(0,y) = \lim_{y \to 0} 0 = 0
  \]
- **y-axis:** \(y=0, x \to 0\):
  \[
  \lim_{x \to 0} g(x,0) = \lim_{x \to 0} \frac{0}{x^4} = 0
  \]
- **Line \(y=x\):**
  \[
  \lim_{x \to 0} g(x,x) = \lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}
  \]

\[
\Rightarrow \lim_{(x,y) \to (0,0)} g(x,y) \text{ DNE}
\]

\[
\text{Example:} \quad \lim_{(x,y) \to (0,0)} \frac{x^3-y^3}{x^2+y^2}
\]

**Use polar coordinates:**
\[
g(x,y) = \frac{r^3(\cos^3 \theta - \sin^3 \theta)}{r^2} = r(\cos^3 \theta - \sin^3 \theta)
\]

\[
|g(u,v)| \leq 2r
\]

So for any \(\varepsilon > 0\), take \(\delta = \varepsilon/2\). Whenever \(|(x,y)| = r < \delta\),

\[
|g(x,y)| \leq 2r \leq 2\delta \leq \varepsilon
\]

So \(\lim_{(x,y) \to (0,0)} g(x,y) = 0\).