Math 231 C,D. Midterm 3, April 24, 2014 TEST F KEY

Name: ____________________  Section Code: _________

Three points will be deducted if these instructions are not followed.

1. Write your full name legibly above.
2. Complete your section code correctly in the boxes above.
3. Code your name and netid correctly on the scantron form.
4. When the exam begins, find your test form on the next page. Code the correct test form on the scantron form.

CDC – WF 10:00-10:50 -Instructor: Menezes, Glen
CDD – WF 11:00-11:50 -Instructor: Menezes, Glen
CDE – WF 12:00-12:50 -Instructor: Mastroeni, Matthew
CDF – WF 1:00-1:50 -Instructor: Butler, Stacey
CDG – WF 2:00-2:50 -Instructor: Butler, Stacey
CDH – WF 3:00-3:50 -Instructor: Mastroeni, Matthew
CDA – WF 8:00-8:50 -Instructor: Orlow, Nathan
CDJ – WF 11:00-11:50 -Instructor: Orlow, Nathan
DDG – WF 2:00-2:50 -Instructor: Jang, Donghoon
DDH – WF 3:00-3:50 -Instructor: Golze, Hiram
DDA – WF 8:00-8:50 -Instructor: Ahmed, Iftikhar
DDF – WF 1:00-1:50 -Instructor: Vellis, Vyron
DDD – WF 11:00-11:50 -Instructor: Heersink, Byron
DDC – WF 10:00-10:50 -Instructor: Heersink, Byron
DDE – WF 12:00-12:50 -Instructor: Golze, Hiram

• The 15 multiple choice answers must be marked on scantron form.
• You must not communicate with other students during this test.
• No written materials of any kind allowed.
• No phones, calculators, iPods or electronic devices of any kind are allowed for ANY reason, including checking the time (you may use a simple wristwatch).
• Do not turn this page until instructed to.
• There are many different versions of this exam.

Violations of academic integrity (in other words, cheating) will be taken extremely seriously, and will be handled under the procedures of Article I, Part 4 of the student code.

Free response scores—for graders only

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Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4 + 100}$.

Mark each of the following True or False. (2 points each)

1. The series converges conditionally.
2. The series converges absolutely.
3. The ratio test is inconclusive.

Consider the series $\sum_{n=0}^{\infty} \frac{(-5)^n}{(n + 5)!}$.

Mark each of the following True or False. (2 points each)

4. The ratio test is inconclusive.
5. The series converges.
6. The series converges absolutely.
7. The series converges by the ratio test.
8. (5 points) Which is the correct definition of the Taylor series of \( g(x) \) centered at \( b \)?

(A) \[ \sum_{n=0}^{\infty} \frac{g^{(n)}(b)}{n!} (x - b)^n \]

(B) \[ \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!} (x - b)^n \]

(C) \[ \sum_{n=1}^{\infty} \frac{g^{(n)}(b)}{n!} (x - b)^n \]

(D) \[ \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} (x - b)^n \]

(E) \[ \sum_{n=1}^{\infty} \frac{g^{(n)}(x)}{n!} (x - b)^n \]

Suppose that the power series \( \sum_{n=0}^{\infty} c_n x^n \) converges when \( x = -2 \) and diverges when \( x = 4 \).

What can be said about the following series? (3 points each)

9. \( \sum_{n=0}^{\infty} c_n (1.2)^n \)

(A) Diverges.

(B) [Converges.]

(C) Impossible to determine with the information given.

10. \( \sum_{n=0}^{\infty} c_n (-4)^n \)

(A) Converges.

(B) Diverges.

(C) [Impossible to determine with the information given.]

11. \( \sum_{n=0}^{\infty} c_n 5^n \)

(A) Impossible to determine with the information given.

(B) Converges.

(C) [Diverges.]


12. (5 points) The first ten terms of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{10}} \) are used to approximate the sum of the series. According to the alternating series estimation theorem, what is the maximum error in this approximation?

(A) \( \frac{1}{10^{10}} \)
(B) \( \frac{3}{10^5} \)
(C) \( \frac{3}{11^4} \)
(D) \( \frac{1}{11^{10}} \)
(E) None of the above.

---

13. (5 points) Find the Maclaurin series for \( x^2 \cos(3x) \).

(A) \( \sum_{n=0}^{\infty} \frac{(-9)^n x^{2n}}{(2n)!} \)
(B) \( \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+3}}{(2n)!} \)
(C) \( \sum_{n=0}^{\infty} \frac{(-9)^n x^{2n}}{(2n)!} \)
(D) \( \sum_{n=0}^{\infty} \frac{g^n x^{2n+2}}{(2n)!} \)
(E) \( \sum_{n=0}^{\infty} \frac{(-9)^n x^{2n+2}}{(2n)!} \)
14. (5 points) Find the first three terms of the Taylor series for \( \ln x \) at \( a = 3 \).

(A) \( 1 + \frac{x}{3} - \frac{x^2}{9} \)

(B) \( \ln 3 + \frac{1}{3}(x - 3) - \frac{1}{9}(x - 3)^2 \)

(C) \( \ln 3 + \frac{1}{3}(x - 3) + \frac{1}{9}(x - 3)^2 \)

(D) \( \ln 3 + \frac{1}{3}(x - 3) - \frac{1}{18}(x - 3)^2 \)

(E) \( \ln 3 - \frac{1}{3}(x - 3) + \frac{1}{18}(x - 3)^2 \)

15. (5 points) Let \( f(x) = -\ln \left(1 - \frac{x}{3}\right) \). You are given that \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n} \) for \( |x| < 1 \).

Use this to evaluate the derivative \( f^{(15)}(0) \).

(A) \( \frac{1}{15 \cdot 3^{15}} \)

(B) \( 15! \)

(C) \( \frac{14!}{3^{15}} \)

(D) \( \frac{1}{14 \cdot 3^{14}} \)

(E) \( \frac{15!}{3^{15}} \)
Free response. Show work and circle answers.

1. (16 points) Give brief and complete justification.
   
   a) Find the radius of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(x-5)^n}{7^n(n+3)} \).

   \[
   \lim_{n \to \infty} \left| \frac{(x-5)^{n+1}}{7^{n+1}(n+4)} \cdot \frac{7^n(n+3)}{(x-5)^n} \right| = \lim_{n \to \infty} \frac{|x-5|}{7} \quad \rightarrow \quad \frac{|x-5|}{7} < 1
   \]

   Solving \( \frac{|x-5|}{7} < 1 \) gives \( |x-5| < 7 \).

   Radius.

   b) Find the interval of convergence of this power series.

   \(-7 < x-5 < 7 \rightarrow\) \(-2 < x < 12\)

   Endpoint check:
   
   \[ x = -2 \Rightarrow \sum_{n=0}^{\infty} \frac{(-7)^n}{7^n(n+3)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3} \text{ convergent by ALT. SIG. TEST} \]

   \[ x = 12 \Rightarrow \sum_{n=0}^{\infty} \frac{7^n}{7^n(n+3)} = \sum_{n=0}^{\infty} \frac{1}{n+3} \text{ Divergent by \( \sum \frac{1}{n} \)} \]

   Then, \([-2, 12)\)
2. (16 points)
a) Use the binomial theorem to find the Maclaurin series for \( f(x) = (1 + x)^{-5} \). Do not simplify the binomial coefficients. Give the radius of convergence.

\[
(1 + x)^{-5} = \sum_{n=0}^{\infty} \binom{-5}{n} x^n, \quad |x| < 1.
\]

b) Use part (a) to write down the second degree Taylor polynomial \( T_2(x) \) for \( f(x) \). You must simplify the binomial coefficients in this part.

\[
T_2(x) = \binom{-5}{2} x^2 - \binom{-5}{1} x + 1
\]

\[
= (-55)x^2 + 15x + 1
\]

\[
T_2(x) = 1 - 55x + 15x^2
\]

c) Estimate the maximum possible error in the approximation \( f(x) \approx T_2(x) \) on \([0, \frac{1}{10}]\).

Since it is alternating, \( |x| < 1 \),

\[
\text{Error} \leq \binom{-5}{3} |x|^3 \text{ on } [0, \frac{1}{10}]
\]

\[
\leq \left| \frac{(-5)(-6)(-7)}{3!} \right| \left( \frac{1}{10} \right)^3
\]

\[
= \frac{143}{5000}
\]
3. (16 points)
a) Write the first four non-zero terms in each MacLaurin series. Simplify these terms.

- \( e^{-5x^2} \)
  \[
  e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies e^{-5x^2} = \sum_{n=0}^{\infty} \frac{(-5x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-5)^n x^{2n}}{n!}
  \]
  \[
  = 1 - 5x^2 + \frac{(-5)^2 x^4}{2!} + \frac{(-5)^3 x^6}{3!} + \ldots
  \]

- \( \int e^{-5x^2} \, dx \)
  \[
  \int e^{-5x^2} \, dx = \int \left[ 1 - 5x^2 + \frac{25}{2} x^4 - \frac{125}{6} x^6 + \ldots \right] \, dx
  \]
  \[
  = C + x - \frac{5x^3}{3} + \frac{5x^5}{5!} - \frac{125x^7}{42} + \ldots
  \]

- \( \frac{\sin x}{x} \)
  \[
  \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots
  \]

b) Use the series in part (a) to evaluate the limit

\[
\lim_{x \to 0} \frac{1}{x^2} \left( e^{-5x^2} - \frac{\sin x}{x} \right).
\]

No credit for using another method.

\[
\lim_{x \to 0} \frac{1}{x^2} \left[ (1 - 5x^2 + \frac{25}{2} x^4 - \frac{125}{6} x^6 + \ldots) - (1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots) \right]
\]
\[
= \lim_{x \to 0} \frac{1}{x^2} \left[ x^2 \left( -5 + \frac{1}{6} \right) + x^4 \left( \frac{25}{2} - \frac{1}{5} \right) + \ldots \right]
\]
\[
= \lim_{x \to 0} \left[ -5 + \frac{1}{6} + x^4 \left( \ldots \right) + x^6 \left( \ldots \right) \right] = -5 + \frac{1}{6} = -\frac{29}{6}
\]
4. (14 points)

a) Find the degree 2 Taylor polynomial $T_2(x)$ for the function $f(x) = x^{5/2}$ centered at $a = 4$.

\[
\begin{align*}
\frac{f(4)}{x^{3/2}} &= 32 \\
\frac{f'(4)}{x^{3/2}} &= \frac{5}{2} \\
\frac{f''(4)}{x^{3/2}} &= \frac{15}{8}
\end{align*}
\]

\[
T_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2
\]

\[
= 32 + 20(x-4) + \frac{15}{4}(x-4)^2
\]

b) Use Taylor's theorem to estimate the maximum error in the approximation $f(x) \approx T_2(x)$ in the range $1 \leq x \leq \frac{3}{2}$.

(You must give a numerical answer, but you do not need to evaluate it.)

\[
|R_2(x)| \leq \frac{f^{(3)}(4)}{3!}(x-4)^3
\]

\[
f^{(3)}(4) = \frac{5}{8}\sqrt{8}
\]

\[
|R_2(x)| \leq \frac{15}{8} \cdot \left(\frac{3}{2}\right)^3
\]

\[
\text{should be squared.}
\]