Math 231 C,D. Midterm 3, April 25, 2014 TEST C KEY

Name: ___________________________  Section Code: □□□

Three points will be deducted if these instructions are not followed.

1. Write your full name legibly above.
2. Complete your section code correctly in the boxes above.
3. Code your name and netid correctly on the scantron form.
4. When the exam begins, find your test form on the next page. Code the correct test form on the scantron form.

CDC   - WF 10:00-10:50 - Instructor: Vellis, Vyron
CDD   - WF 11:00-11:50 - Instructor: Menezes, Glen
CDE   - WF 12:00-12:50 - Instructor: Mastroeni, Matthew
CDF   - WF 1:00-1:50 - Instructor: Butler, Stacey
CDG   - WF 2:00-2:50 - Instructor: Butler, Stacey
CDH   - WF 3:00-3:50 - Instructor: Mastroeni, Matthew
CDA   - WF 8:00-8:50 - Instructor: Orlow, Nathan
CDJ   - WF 11:00-11:50 - Instructor: Orlow, Nathan
DDG   - WF 2:00-2:50 - Instructor: Jang, Donghoon
DDH   - WF 3:00-3:50 - Instructor: Golze, Hiram
DDA   - WF 8:00-8:50 - Instructor: Ahmed, Iftikhar
DDF   - WF 1:00-1:50 - Instructor: Vellis, Vyron
DDD   - WF 11:00-11:50 - Instructor: Heersink, Byron
DDC   - WF 10:00-10:50 - Instructor: Heersink, Byron
DDE   - WF 12:00-12:50 - Instructor: Golze, Hiram

- The multiple choice answers must be marked on scantron form.
- You must not communicate with other students during this test.
- No written materials of any kind allowed.
- No phones, calculators, iPods or electronic devices of any kind are allowed for ANY reason, including checking the time (you may use a simple wristwatch).
- Do not turn this page until instructed to.
- There are many different versions of this exam.

Violations of academic integrity (in other words, cheating) will be taken extremely seriously, and will be handled under the procedures of Article 1, Part 4 of the student code.

Free response scores—for graders only

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Multiple choice. Mark answers on scantron form.

Your test form is C. Code this on the scantron form now.

Consider the series \( \sum_{n=0}^{\infty} \frac{(-5)^n}{(n+5)!} \).

Mark each of the following True (A) or False (B). (2 points each)

1. The ratio test is inconclusive.
2. The series converges.
3. The series converges absolutely.
4. The series converges by the ratio test.

Consider the series \( \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4 + 100} \).

Mark each of the following True or False. (2 points each)

5. The series converges conditionally.
6. The series converges absolutely.
7. The ratio test is inconclusive.
8. (3 points) Which is the correct definition of the Taylor series of \( g(x) \) centered at \( b \)?

(A) \[ \sum_{n=0}^{\infty} \frac{g^{(n)}(b)}{n!} (x - b)^n \]

(B) \[ \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!} (x - b)^n \]

(C) \[ \sum_{n=1}^{\infty} \frac{g^{(n)}(b)}{n!} (x - b)^n \]

(D) \[ \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} (x - b)^n \]

(E) \[ \sum_{n=1}^{\infty} \frac{g^{(n)}(x)}{n!} (x - b)^n \]

Suppose that the power series \( \sum_{n=0}^{\infty} c_n x^n \) converges when \( x = -3 \) and diverges when \( x = 5 \). What can be said about the following series? (3 points each)

9. \[ \sum_{n=0}^{\infty} c_n (-2.2)^n \]

(A) Diverges.

(B) [Converges.]

(C) Impossible to determine with the information given.

10. \[ \sum_{n=0}^{\infty} c_n (-5)^n \]

(A) Converges.

(B) Diverges.

(C) [Impossible to determine with the information given.]

11. \[ \sum_{n=0}^{\infty} c_n (-3.5)^n \]

(A) [Impossible to determine with the information given]

(B) Converges.

(C) Diverges.
12. (5 points) Find the Maclaurin series for \( x^2 \cos(5x) \).

(A) \[ \sum_{n=0}^{\infty} \frac{(-25)^n x^{2n}}{(2n)!} \]

(B) \[ \sum_{n=0}^{\infty} \frac{(-5)^n x^{2n+3}}{(2n)!} \]

(C) \[ \sum_{n=0}^{\infty} \frac{(-25)^n x^{2n}}{(2n)!} \]

(D) \[ \sum_{n=0}^{\infty} \frac{25^n x^{2n+2}}{(2n)!} \]

(E) \[ \sum_{n=0}^{\infty} \frac{(-25)^n x^{2n+2}}{(2n)!} \]

13. (5 points) The first nine terms of the series \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{12}} \] are used to approximate the sum of the series. According to the alternating series estimation theorem, what is the maximum error in this approximation?

(A) \[ \frac{1}{10^{12}} \]

(B) \[ \frac{3}{10^{12}} \]

(C) \[ \frac{3}{11^{12}} \]

(D) \[ \frac{1}{11^{12}} \]

(E) None of the above.

14. (5 points) Let \( f(x) = -\ln \left(1 - \frac{x}{3}\right) \). You are given that \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n} \) for \( |x| < 1 \).

Use this to evaluate the derivative \( f^{(15)}(0) \).

(A) \[ \frac{1}{15 \cdot 3^{15}} \]

(B) \[ 15! \]

(C) \[ \frac{14!}{3^{15}} \]

(D) \[ \frac{1}{14 \cdot 3^{14}} \]

(E) \[ \frac{15!}{3^{15}} \]
15. (5 points) Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(x - 5)^n}{6^n(n + 1)} \).

(A) \([-6, 6]\)
(B) \((-1/6, 1/6]\)
(C) \([-1, 11]\)
(D) \((-29/6, 31/6]\)
(E) \((-1, 11]\)

16. (4 points) Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{n!3^n(x + 1)^n}{(2n)!} \).

(A) \([-4, 2]\)
(B) \([-4/3, -2/3]\)
(C) \([-2, 4]\)
(D) \([-4/3, -2/3]\)
(E) \((-\infty, \infty)\)
Free response. Show work and circle answers.

1. (16 points) Give brief and complete justification.
   
a) Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{7^n(n^2+5)}$.

   \[
   \lim_{n \to \infty} \left| \frac{\frac{(x-5)^{n+1}}{7^{n+1}(n+1)^2+5}}{\frac{(x-5)^n}{7^n(n^2+5)}} \right| = \lim_{n \to \infty} \frac{|x-5|}{7} < 1 \Rightarrow |x-5| < \left(\frac{7}{1}\right) = 7
   \]

   Radius

b) Find the interval of convergence of this power series.

\[-7 < x < 5 \Rightarrow -2 < x < 12\]

Check endpoints:

at $x = -2$, $\sum \frac{(-7)^n}{7^n(n^2+5)} = \sum \frac{(-1)^n}{n^2+5}$ convergent by alternating series test

at $x = 12$, $\sum \frac{7^n}{7^n(n^2+5)} = \sum \frac{1}{n^2+5}$ convergent by LCT with $\sum \frac{1}{n^2}$

$\therefore [-2, 12]$
(c) (3 points) What is the 6th derivative of \((1 + 5x)^{-3}\) at 0? (No credit for just computing out the 6 derivatives!)

First, start the series at \(n = 0\)

\[
\frac{1}{50} \sum_{n=2}^{\infty} \left( -1 \right)^n n(n-1) \frac{n}{5} x^{n-2}
\]

\[
= \frac{1}{50} \sum_{n=0}^{\infty} \left( -1 \right)^{n+2} (n+2)(n+1) \frac{n+2}{5} x^n
\]

\[
= \frac{f^{(6)}(0)}{50} \frac{1}{n!}
\]

\[
f^{(6)}(0) = \left[ (-1)^7 \cdot (8) \cdot (7) \cdot 5^8 \right] \cdot 50
\]

\[
\Rightarrow f^{(6)}(0) = -8 \cdot 7 \cdot 5^8 \cdot 6!
\]

Secondary way: \(3\) \((1 + 5x)^{-3} = \sum_{n=0}^{\infty} \left( -\frac{3}{5} \right)^n (5x)^n = \sum_{n=0}^{\infty} \left( -\frac{3}{5} \right)^n 5^n x^n
\]

\[
f^{(n)}(5) = \left( -\frac{3}{5} \right)^n 5^n \Rightarrow f^{(6)}(0) = 6! \cdot 5^6 \cdot (-3) \cdot (4) \cdot (5) \cdot (6) \cdot (7)\]

\[
= \frac{56 \cdot 8!}{6!}
\]
2. (a) (8 points) RECOMPUTE the MacLaurin series for \( \frac{1}{(1+5x)^3} \) USING THE FACT that it is the second derivative of \( \frac{1}{50(1+5x)} \):

\[
\left[ \frac{1}{50} (1+5x)^{-1} \right]' = -\frac{1}{10} (1+5x)^{-2}
\]

\[
\left[ \frac{1}{50} (1+5x)^{-1} \right]'' = \frac{1}{5} \cdot 5 \cdot (1+5x)^{-3} = \frac{1}{(1+5x)^3}
\]

\[
\frac{1}{50} \left( \frac{1}{5x+1} \right) = \frac{1}{50} \sum_{n=0}^{\infty} (-1)^n (5x)^n = \frac{1}{50} \sum_{n=0}^{\infty} (-1)^n 5^nx^n
\]

\[
\left[ \frac{1}{50} \frac{1}{5x+1} \right]' = \frac{1}{50} \sum_{n=1}^{\infty} (-1)^n 5n (5x)^{n-1} = \frac{1}{50} \sum_{n=1}^{\infty} (-1)^n 5nx^n
\]

\[
\left[ \frac{1}{50} \frac{1}{5x+1} \right]'' = \frac{1}{(1+5x)^3} = \frac{1}{50} \sum_{n=2}^{\infty} (-1)^n n(n-1) 5^nx^{n-2}
\]

(b) (3 points) What is the radius of convergence of this series?

\[ |5x| < 1 \implies |x| < \frac{1}{5} \]
3. (16 points)

a) Write the first four non-zero terms in each MacLaurin series. Simplify these terms.

- \( \sin(3x) \)
  \[
  \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
  \sin(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (3x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n+1} = 3x - \frac{9}{3!} x^3 + \frac{81}{5!} x^5 - \frac{729}{7!} x^7 + \ldots
  \]

- \( x e^{x^2} \)
  \[
  e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
  e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \\
  x \cdot e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!} = x + \frac{4}{2!} x^3 + \frac{7}{3!} x^5 + \frac{10}{4!} x^7 + \ldots
  \]

- \( \int xe^{x^2} \, dx \)
  \[
  \int \left[ x + \frac{x^3}{2!} + \frac{x^5}{3!} + \ldots \right] \, dx = \frac{x^2}{2} + \frac{x^5}{5!} + \frac{x^8}{8 \cdot 2!} + \frac{x^{11}}{11 \cdot 3!} + \ldots
  \]

b) Use the series in part (a) to evaluate the limit

\[
\lim_{x \to 0} \left( \frac{\sin(3x) - xe^{x^2}}{x} \right)
\]

No credit for using another method.
4. (14 points)

a) Find the degree 2 Taylor polynomial \( T_2(x) \) for the function \( f(x) = x^{7/2} \) centered at \( a = 4 \).

\[
\begin{align*}
  f(x) &= x^{7/2} \Rightarrow f(4) = 4^{7/2} = 64 \\
  f'(x) &= \frac{7}{2} x^{5/2} \Rightarrow f'(4) = 7 \cdot 16 = 112 \\
  f''(x) &= \frac{7 \cdot 5}{2} \cdot x^{3/2} \Rightarrow f''(4) = 70 \\
  T_2(x) &= 64 + 112 \cdot (x-4) + \frac{70}{2!} \cdot (x-4)^2.
\end{align*}
\]

b) Use Taylor's theorem to estimate the maximum error in the approximation \( f(x) \approx T_2(x) \) in the range \( 1 \leq x \leq \frac{3}{2} \).

(You must give a numerical answer, but you do not need to evaluate it.)

\[
\begin{align*}
  |R_2(x)| &\leq \frac{M}{3!} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \\
  |R_2(x)| &= \left| \frac{f'''(\xi)}{3!} \cdot (x-4)^3 \right| \\
  &\leq \frac{7 \cdot 5 \cdot 3}{8} \cdot \left( \frac{3}{2} \right)^{1/2} \cdot |(x-4)^3| \\
  &\leq \frac{7 \cdot 5 \cdot 3 \sqrt{3}}{8 \sqrt{2}} \cdot (3)^{3}.
\end{align*}
\]