Thus: Let \( f \) be analytic everywhere inside \( \Delta z \)
S.C. counterclockwise.

If \( z_0 \) is any point interior to \( C \),

\[
J(z_0) = \frac{1}{2\pi i} \oint \frac{f(z_1) \, dz}{z-z_0}.
\]

ie, the values of analytic \( f \) are completely

\( \text{det} \) by values of \( f \) \( \text{on} \) \( C \).

\[\begin{align*}
\Delta \Omega; \quad \oint \frac{f(z)}{z-z_0} \, dz &= \oint \frac{f(z)}{z-z_0} \, dz, \\
\text{Take } C_0: & \quad \oint \frac{f(z)}{z-z_0} \, dz = \oint \frac{f(z)}{z-z_0} \, dz, \\
\text{Thus, } & \quad \frac{f(z)}{z-z_0} \text{ is analytic on } \gamma_0, C \text{, and in the region 0 the-}
\end{align*}\]
Here: \[ \oint \frac{dz}{z-\pi} = 2\pi i. \]

\[ G: \pi^2 = 2\pi i. \]

\[ dt: = \rho \cdot e^{i\theta} \cdot d\theta \]

\[ = 1 \int \frac{f(t)}{z-t} \, dz = (2\pi i) f(\pi) = \oint \frac{f(z) - f(\pi)}{z-\pi} \, dz. \]

Sine \( f \) is analytic, \( -i \) continuous.

for \( \Re z \gt 0, \) \( \exists \delta \) s.t.

\[ |f(z) - f(\pi)| < \varepsilon \quad \text{where} \quad |z-\pi| < \delta. \]

Let \( \rho < \delta. \) \( \Rightarrow \) \( |z-\pi| < \rho < \delta. \) \( \therefore \)

\[ \left| \int \frac{f(z) - f(\pi)}{z-\pi} \, dz \right| < \frac{\varepsilon}{\rho} \cdot 2\pi \rho. \]

So: \[ \left| \oint \frac{f(z)}{z-\pi} \, dz - 2\pi i f(\pi) \right| < 2\pi \varepsilon. \]
**EX:** \( C: \) pos. oriented circle \(|z| = 1\) about the origin.

\( f(z) = \frac{\cos t}{z^2 + 9} \) analytic.

\[
\int_{C} \frac{\cos t}{t(z^2 + 9)} \, dz = \int_{C} \frac{\cos t}{2t} \, dz = 2\pi i f(0) = \frac{2\pi i}{9}
\]
Extended CIF: \( \text{let } f \text{ be analytic} \)

\[
\frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{1}{2\pi i} \int_C \frac{f(s)}{s(t+\Delta t) - s(t-\Delta t)} ds
\]

Let \( d = \text{small dist. between } t \text{ and } C \) and assume \( 0 < |\Delta t| < d \).

\[
\frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{1}{2\pi i} \int_C \frac{1}{s(t+\Delta t) - s(t-\Delta t)} ds
\]

\[
= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-t+\Delta t)(s-t-\Delta t)} ds
\]

\[
= \frac{1}{(s-t)^2} \frac{\Delta(s-t)}{(s-t-\Delta t)(s-t+\Delta t)} = \frac{s-t+\Delta t-\Delta z}{(s-t-\Delta t)(s-t+\Delta t)^2}
\]

\[
= \frac{1}{(s-t)^2} + \frac{\Delta z}{(s-t-\Delta t)(s-t)^2}
\]

\( |s-t| \geq d \) \& \( |\Delta t| < d \) so:

\[
|s-t-\Delta t| \geq |s-t| - |\Delta t| \geq d - |\Delta t| > 0
\]
\[ \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = -\frac{1}{2\pi i} \int_C \frac{f(s)}{(s-t)^2} ds = 0 \]

So:
\[ f'(t) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-t)^2} ds \]

So:

Thm: Extended C.I.F.

Let \( f \) be analytic inside and on a simple closed contour \( C \), taken in positive sense.

If \( z_0 \) is any pt interior to \( C \), then:

\[ f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n=0, 1, 2, \ldots \]
If $C$ is positively oriented unit circle $|z| = 1$, and $f(1) = \frac{2\pi}{3}$

$$\oint_{|z|=1} \frac{2\pi}{3z} \, dz = f''(1) = \frac{8\pi i}{3} \cdot f''(1) = \frac{8\pi i}{3}.$$

**Some consequences:**

There 1. If $f$ is analytic at a given point, its derivatives of all orders are so, too.

2. Let $f$ be analytic at $z_0$. Then:

$$f''(z) = \frac{2}{2\pi i} \oint_{|z|=\epsilon} \frac{f(z)}{(z-z_0)^3} \, ds$$

Since $f''$ exist in this nbhd $f'$ is analytic in this nbhd.

**Note:** Apply this to calculate $f'''$ and $f^{(4)}$. 

**Example:**

Let $C_0$ be a unit circle with radius $\epsilon/2$.
**Corollary** If \( f(z) = u(x,y) + i v(x,y) \) is analytic at a point \( z = (x,y) \), then its components \( u \) & \( v \) have continuous partial derivatives of all orders at that point.

**Bec:** If \( f \) is analytic, \( \Rightarrow f', f'' \text{ etc.} \) cont.
\[ \Rightarrow u_x, u_y \text{ are continuous} \]
Also since \( f'' \) is analytic, \( u_{xx}, u_{xy}, u_{yy} \) are cont.

**THEOREM** Let \( f \) be continuous on \( \partial \Omega \) (the boundary).

If \( f \) has no poles on \( \partial \Omega \),

in every closed contour \( \Gamma \subset \Omega \), then \( f \) is analytic throughout \( \Omega \).

**Bec:** Use hyp. 3 satisfied, \( f \) analytic outside.

If \( Ef + F'h = F'h \) \& \( F \) analytic,

By Thm 1, \( F' \) is also analytic. \( \Rightarrow f \) is analytic.
Theorem 3 \hspace{1cm} Suppose that \( f \) is analytic inside and on a positively oriented circle \( C \), centered at \( z_0 \) with radius \( R \).

If \( M_R \) denotes the max. value of \( |f(z)| \) on \( C \),

\[
|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n} \quad (n = 1, 2, \ldots)
\]

---

Cauchy's Inequality.

\[
|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_{C} \frac{f(z) \, dz}{(z-z_0)^{n+1}} \right| \leq \frac{n! M_R}{2\pi R^{n+1}} \cdot 2\pi R
\]