Continuity:

- $f$ is continuous at a point $x_0$, if:
  1. $\lim_{t \to x_0} t$ exists.
  2. $f(t)$ exists.
  3. $\lim_{t \to x_0} f(t) = f(x_0)$. 

Composition of cont. func is cont. If $f(t)$ is cont. and non-zero at $x_0$, then $f(t) \neq 0$ throughout some neighborhood of $x_0$ point.

Pf: Assume $f(x_0) \neq 0$, and let $\varepsilon = \frac{|f(x_0)|}{2}$.

By continuity, there is $\delta$ such that $|f(t) - f(x_0)| < \frac{|f(x_0)|}{2}$ whenever $|t - x_0| < \delta$.

But if $f(t) = 0$, in the given neighborhood,

Then: $|f(x_0)| < \frac{|f(x_0)|}{2}$, which is not possible.
THEM: Components of $f$ are continuous $\iff f$ is continuous.

THM 4: If a func $f$ is continuous throughout a region $\mathbb{R}$ that is closed and bdd, then

$$|f(x)| \leq M$$

where equality holds for at least one x each $E$.

BCF: Let $f(x) = u + iv$. continuous

$$u \& v$$ are cont. func.

$$|f(x)| = \sqrt{u^2 + v^2}$$ is continuous

$$|f(x)|$$ takes its max. on a closed & bdd set.
**DIFFERENTIALS:**

Let \( f \) be a function whose domain includes \( 12 - 20 \) < \( 3 \).

**Derivative of \( f \) at \( x \) is:**

\[
 f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{\Delta z \to 0} \frac{f(x + \Delta z) - f(x)}{\Delta z}.
\]

\( f \) is differentiable at \( x \) when \( f'(x) \) exists.

**Notation:**

\[ \Delta t = t - x, \quad \Delta w' = f(t + \Delta z) - f(t), \]

\[
 \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.
\]

**Example:**

\[
 f(12) = \frac{1}{2}. \quad \text{At each } x = 12,
\]

\[
 \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{2} \left( \frac{1}{x + \Delta x} - \frac{1}{x} \right) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} \left( \frac{1}{x + \Delta x} - \frac{1}{x} \right) = -\frac{1}{x^2}.
\]
\[ f(\tau) = \bar{z}, \]
\[ \Delta w = \frac{\bar{z} + \Delta z - \bar{z}}{\Delta z} = \frac{\Delta z}{\Delta z} \]
\[ \Delta \tau = (\Delta x, 0) \quad \text{will give different} \]
\[ \Delta \tau = (0, \Delta y) \quad \text{limits} \]

\[ \therefore \]

\[ = \text{describe DNE} \]

\[ f(\tau) = |z|^2 \]
\[ \Delta w = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\bar{z} + \bar{\Delta z}) - 4\bar{z} \bar{z}}{\Delta z} \]
\[ \quad \quad \quad = \frac{z\bar{z} + z\bar{\Delta z} + (\Delta z)\bar{z} + \Delta z \bar{z} - 2\bar{z}}{\Delta z} \]

\[ \quad \quad \quad = \bar{z} + \Delta \bar{z} + \frac{\Delta z}{\Delta z} \]

Horizontal approach: \( \Delta \bar{z} = \Delta \tau \)
Vertical approach: \( \Delta z = -\Delta \bar{z} \)

\[ \therefore \quad \frac{\Delta w}{\Delta \tau} = \bar{z} \quad \text{and} \quad \frac{\Delta w}{\Delta z} = 2 - \bar{z} - \tau \]

\[ \lim_{\Delta \tau \to 0} \frac{\Delta w}{\Delta \tau} = \bar{z} + z = \bar{z} - \tau \quad (\Rightarrow \tau = 0) \]
\[ \frac{du}{dt} \text{ exists at } t=0, \text{ because,} \]

when \( t=0, \) \[ \frac{\Delta u}{\Delta t} = \frac{\Delta f}{\Delta z} \]

\[ \text{Outcomes of the Example} \]

1. \( f = u + iv \) can be differentiable at a point \( z=(x,y) \) but nowhere else in any neighborhood of that point.

2. \( f(z) = 1 + z^2 \Rightarrow u(x,y) = x^2 + y^2 \]
   \[ v(x,y) = 0. \]

Conversely, if \( u \) and \( v \) have continuous partial derivatives of all orders at all points \( z=(x,y) \), but \( f \) may not be differentiable there.

3. \( \text{If } u \text{ and } v \text{ are continuous } \Rightarrow f \text{ is continuous} \]
   \[ \Rightarrow f \text{ is differentiable} \]

But \( \frac{du}{dt} \text{ exists at } c \text{ pt. } \Rightarrow f \text{ is continuous at } c \text{ pt. } \]