A set of points \( z = (x, y) \) in \( \mathbb{C} \) is called an arc if

\[
x = x(t), \quad y = y(t), \quad a \leq t \leq b
\]

\( x \) and \( y \) are continuous!

- A simple arc (Jordan arc) is an arc which does not intersect itself \((t_1 \neq t_2 \implies z(t_1) \neq z(t_2))\)

**Ex:**

\[
x + i \cdot \frac{2 + 1}{4} \quad 0 \leq x \leq 1
\]

\[
x + i, \quad 0 \leq x \leq 2
\]

is a simple arc.

- \( z = e^{i \theta}, \quad 0 \leq \theta \leq 2\pi \) a circle,
- \( z = e^{-i \theta}, \quad 0 \leq \theta \leq 2\pi \) circle, oriented clockwise
- \( z = e^{i \theta}, \quad 0 \leq \theta \leq 2\pi \) traverses circle twice.
- \( z = e^{2i \theta}, \quad 0 \leq \theta \leq 2\pi \) traverses circle twice.
1. If \( z(t) = x(t) + iy(t) \), \( z'(t) = x'(t) + iy'(t) \) |
\[
\left| z'(t) \right| = \sqrt{(x'(t))^2 + (y'(t))^2}
\]
on \( a \leq t \leq b \).

\[
L = \int_a^b \left| z'(t) \right| \, dt
\]

let \( T = \frac{z'(t)}{\left| z'(t) \right|} \) (unit tangent vector).

* A smooth arc is an arc with a continuous \( T(t+\delta t) \).

(\( \delta t \); we want \( z'(t) \) to be continuous)

* Contour = a piecewise smooth arc.
Contour integrals

Defined in terms of values of $f(z(t))$, along contour $C$, from a pt $z=a$ to $z=b$.

It is a line integral.
Its value depends on $f$ and on $C$. Written $\int_{z=a}^{z=b} f(z) \, dz$.

Let $z = z(t)$, $a \leq t \leq b$ be the contour $C$ from $z = z(a)$ to $z = z(b)$.

As we assume that $f(z(t))$ is p.w. continuous on $a \leq t \leq b$, ($\equiv f(z)$ is p.w. continuous on $C$)

Cont. integral in terms of parameter $t$ will be:

$\int_{C} f(t) \, dz = \int_{a}^{b} f(z(t)) \cdot z'(t) \, dt$.

Since $C$ is contour, $z'(t) \neq 0$
on $(a,b)$, and p.w cont. on $[a,b]$.

- $C$ is same set of pts, but the order of those with order reversed.
Some properties:

1. If \( C \) has representation \( a \leq t \leq b \), \( z = z(t) \),
   \(-C\) has rep. \( -b \leq t \leq -a \), \( z = z(-t) \).

2. \[
   \int_{-a}^{a} f(t) \, dt = \int_{-b}^{b} f(z(t)) \cdot z'(t) \, dt
   \]
   \( = \int_{a}^{b} f(z(t)) \cdot z'(t) \, dt \)
   \( = -\int_{a}^{b} f(z(t)) \cdot z'(t) \, dt \)
   \( = -\int_{a}^{b} f(t) \, dt, \)

3. \[
   \int_{a}^{b} |f| \, dt = \int_{a}^{b} f(t) \, dt + \int_{a}^{b} |f(t)| \, dt
   \]
   \( = \int_{a}^{b} f(t) \, dt \cdot z(t) \, dt + \int_{a}^{b} g(t) \, dt \cdot z(t) \, dt \)
   \( = \int_{a}^{b} f(t) \, dt - \int_{a}^{b} f(t) \, dt \)
   \( = \int_{a}^{b} f(t) \, dt \cdot z(t) \, dt + \int_{a}^{b} g(t) \, dt \cdot z(t) \, dt \)

4. \[ C = C_1 \cup C_2 \]
   \( C_1: a \leq t \leq c_1, \quad C_2: c_2 \leq t \leq b \)
   \( \int_{C} \cdot f(t) \, dt = \int_{C_1} f(t) \, dt + \int_{C_2} f(t) \, dt. \)
Evaluating

\[ \int f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz \]

\[ f(z) = y - x - i3x^2 \quad (z = x + iy) \]

- OA leg: \( x = 0 \) & \( 0 \leq y \leq 1 \). \( z = iy = \int_{0}^{1} dy \)

\[ \int_{OA} f(z)dz = \int_{0}^{1} y^2 \frac{1}{2} = \frac{1}{4} \]

- AB leg: \( y = 1 \), \( 0 \leq x \leq 1 \).

\[ \int_{AB} f(z)dz = \int_{0}^{1} (1 - x - 3i x^2) \]

\[ = \left( x - \frac{x^2}{2} - \frac{3ix^3}{3} \right) \bigg|_{0}^{1} = 1 - \frac{1}{2} - i \]

\[ c = c_1 - c_2 \]

\[ 0 \leq \theta \leq \pi \]
\[ \oint_{C_2} dz = \int_0^1 (-1)^2 \cdot (1+i) \, dx = \int_0^1 (-3ix^2 + 3x^2) \, dx \]

\[ z = i + ix \]
\[ dz = (1+i) \, dx \]
\[ = -\frac{3i}{7} x^3 + \frac{3}{7} x^2 \bigg|_0^1 \]
\[ = 1-i \]

**Observe:** \[ I_1 - I_2 = \frac{1-i}{2} \neq 0. \]
$C: z = 3e^{i \theta}, \ 0 \leq \theta \leq \Pi.$

from $z = 3$ to $z = -3.$

$\int = \int_c z^{\frac{1}{2}} \, dz \ = \ ?$

for $z^{\frac{1}{2}} = \exp \left( \frac{1}{2} \log z \right), \ \text{for } |z| > 0, \ 0 < \arg z < 2\Pi.$

$f(z)$ is not defined at $z = 3.$ ($\theta = 0$ true).

But: $z(\theta) = 3e^{i \theta}$

$f(z(\theta)) = \exp \left( \frac{1}{2} \log(3e^{i \theta}) \right)$

$= \exp \left[ \frac{1}{2} (\ln 5 + i \theta) \right]$

$= \sqrt{3} \ e^{i \theta/2}$

$f(z(\theta))$ is p.w. continuous on $C,$ b/c:

$f(z(\theta)). \ \frac{dz}{d\theta} = -3\sqrt{3} \sin \frac{\theta}{2} + i \cdot 3\sqrt{3} \cos \frac{3\theta}{2}$

When $\theta \to 0^+, \ h \to i.3\sqrt{3}$

$\Rightarrow \ h$ is continuous on $0 \leq \theta \leq \Pi.$

Just define its value at $\theta = 0$ at $i.3\sqrt{3}.$
Then:
\[ I = 3\sqrt{2}i \int_0^{\pi/2} \sin\theta \, d\theta \]
\[ = 3\sqrt{2}i \left. \frac{\cos \theta}{\sqrt{2}} \right|_0^{\pi/2} \frac{\pi}{2} \]
\[ = 3\sqrt{2} \left( \frac{1}{2} \sqrt{2} - 0 \right) \]
\[ = 2\sqrt{3} \left( e^{i\pi/2} - 1 \right) \]
\[ = -2\sqrt{3}i + 2\sqrt{3}i \]

\[ e = \cos \frac{\pi}{2} \]

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