1. Consider the region $D$ in $\mathbb{R}^3$ bounded by the $xy$-plane and the surface $x^2 + y^2 + z = 1$.

(a) Make a sketch of $D$.

**Solution.** The sketch of $D$ is shown below.

(b) The boundary of $D$, denoted $\partial D$, has two parts: the curved top $S_1$ and the flat bottom $S_2$. Parameterize $S_1$ and calculate the flux of $F = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field. Check your answer with the instructor.

**Solution.** To parametrize $S_1$, one has

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 - u^2 \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$  

In order to calculate the flux, first we have

$$\mathbf{r}_u = \langle \cos v, \sin v, -2u \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle,$$

and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle.$$

Therefore, the flux of $F = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field is

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^1 \mathbf{F}(u, v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^1 (0, 0, 1 - u^2) \cdot (2u^2 \cos v, 2u^2 \sin v, u) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^1 (1 - u^2)u \, du \, dv = \frac{\pi}{2}.$$
(c) Without doing the full calculation, determine the flux of $F$ through $S_2$ with the downward pointing normals.

**Solution.** Since $F = 0$ on $S_2$, we know the flux of $F$ through $S_2$ is

$$\iint_{S_2} F \cdot n \, dS = \iint_{S_2} 0 \cdot n \, dS = 0.$$  

(d) Determine the flux of $F$ through $\partial D$ with the outward pointing normals.

**Solution.** By adding up the result from (a) and (b), one gets

$$\iint_{\partial D} F \cdot n \, dS = \iint_{S_1} F \cdot n \, dS + \iint_{S_2} F \cdot n \, dS = \frac{\pi}{2}.$$ 

(e) Apply the Divergence Theorem and your answer in (d) to find the volume of $D$. Check your answer with the instructor.

**Solution.** By the Divergence Theorem, one has

$$\iint_{\partial D} F \cdot n \, dS = \iiint_D \text{div} F \, dV.$$ 

Since $\text{div} F = 1$, one gets

$$\text{Volume}(D) = \iiint_D 1 \, dV = \iiint_D \text{div} F \, dV = \iiint_{\partial D} F \cdot n \, dS = \frac{\pi}{2}.$$ 

2. Consider the vector field $F = (-y, x, z)$.

(a) Compute curl $F$.

**Solution.**

$$\text{curl } F = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{pmatrix} = (0, 0, 2).$$

(b) For the surface $S_1$ above, evaluate $\iint_{S_1} (\text{curl } F) \cdot n \, dA$.

**Solution.** By applying the parametrization of $S_1$ in 1(b), one gets

$$\iint_{S_1} (\text{curl } F) \cdot n \, dA = \int_0^2 \int_0^1 (\text{curl } F) \cdot (r_u \times r_v) \, dudv$$

$$= \int_0^2 \int_0^1 2u \, dudv = 2\pi.$$
(c) Check your answer in (b) using Stokes’ Theorem.

**Solution.** The boundary of \( S_1 \) is a unit circle centered at the origin in the \( xy \)-plane. So we can parametrize it as

\[
C : \mathbf{r}(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi.
\]

Thus, by Stokes’ Theorem, one has

\[
\iint_{S_1} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \int_{0}^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) \, dt = \int_{0}^{2\pi} 1 \, dt = 2\pi.
\]

3. If time remains:

(a) Check your answer in 1(e) by directly calculating the volume of \( D \).

**Solution.** One can use the polar coordinate to calculate the volume of \( D \) in 1(e). Let

\[
x = r \cos \theta, \; y = r \sin \theta, \quad 0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi.
\]

Then

\[
\text{Volume}(D) = \iiint_{x^2 + y^2 \leq 1} (1 - x^2 - y^2) \, dA = \int_{0}^{2\pi} \int_{0}^{1} (1 - r^2) \, r \, dr \, d\theta = \frac{\pi}{2}.
\]

(b) Repeat 2 (b-c) for the surface \( S_2 \) and also for the surface \( \partial D \). What exactly does 2(c) mean for the surface \( \partial D \)?

**Solution.** The normal vector of \( S_2 \) pointing downward is \( \mathbf{n} = -\mathbf{k} \). Thus,

\[
\iint_{S_2} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = \iint_{S_2} (0,0,2) \cdot (0,0,-1) \, dA = -2 \iint_{S_2} \, dA = -2\pi.
\]

To check the above answer using Stokes’ Theorem, one needs the parametrization of the boundary of \( S_2 \). Notice that this boundary is the same as that of \( S_1 \) except the orientation. The boundary of \( S_2 \) is parametrized by

\[
C' : \mathbf{r}(t) = (\cos(2\pi - t), \sin(2\pi - t), 0), \quad 0 \leq t \leq 2\pi.
\]

Thus, by Stokes’ Theorem, one has

\[
\iint_{S_2} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_{0}^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \int_{0}^{2\pi} (-\sin(2\pi - t), \cos(2\pi - t), 0) \cdot (\sin(2\pi - t), -\cos(2\pi - t), 0) \, dt
\]

\[
= \int_{0}^{2\pi} -1 \, dt = -2\pi.
\]
By adding up the two integrals one gets
\[
\iint_{\partial D} (\text{curl} \, F) \cdot \mathbf{n} \, dA = \iint_{S_1} (\text{curl} \, F) \cdot \mathbf{n} \, dA + \iint_{S_2} (\text{curl} \, F) \cdot \mathbf{n} \, dA = 2\pi + (-2\pi) = 0.
\]

Since \( \partial D \) is a surface without any curve boundary, then \( \text{2(c)} \) shows that the integral of \( F \) along the curve boundary of the surface \( \partial D \) must be 0.

(c) For the vector field \( F = (-y, x, z) \) from the second problem, compute \( \text{div(\text{curl} F)} \). Now suppose \( F = (F_1, F_2, F_3) \) is an arbitrary vector field. Can you say anything about the function \( \text{div(\text{curl} F)} \)?

**Solution.** We already know, in \( \text{2(a)} \), that \( \text{curl} \, F = (0, 0, 2) \), so \( \text{div(\text{curl} F)} = 0 \). Generally, suppose \( F = (F_1, F_2, F_3) \) is an arbitrary vector field. Then

\[
\text{curl} \, F = \text{det} \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right),
\]

and

\[
\text{div} \, (\text{curl} \, F) = \partial_x \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) + \partial_y \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) + \partial_z \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.
\]