Curves and integration.

1. Consider the curve $C$ in $\mathbb{R}^3$ given by

$$r(t) = (e^t \cos t) \mathbf{i} + 2 \mathbf{j} + (e^t \sin t) \mathbf{k}$$

(a) Draw a sketch of $C$.

(b) Calculate the arc length function $s(t)$, which gives the length of the segment of $C$ between $r(0)$ and $r(t)$ as a function of the time $t$ for all $t \geq 0$. Check your answer with the instructor.

(c) Now invert this function to find the inverse function $t(s)$. This gives time as a function of arclength, that is, tells how long you must travel to go a certain distance.

(d) Suppose $h: \mathbb{R} \to \mathbb{R}$ is a function. We can get another parameterization of $C$ by considering the composition

$$f(s) = r(h(s))$$

This is called a reparametrization. Find a choice of $h$ so that

i. $f(0) = r(0)$

ii. The length of the segment of $C$ between $f(0)$ and $f(s)$ is $s$. (This is called parametrizing by arc length.)

Check your answer with the instructor.

(e) Without calculating anything, what is $|f'(s)|$?

2. Consider the curve $C$ given by the parametrization $r: \mathbb{R} \to \mathbb{R}^3$ where $r(t) = (\sin t, \cos t, \sin^2 t)$.

(a) Show that $C$ is in the intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$.

(b) Use (a) to help you sketch the curve $C$.

3. (a) Sketch the top half of the sphere $x^2 + y^2 + z^2 = 5$. Check that $P = (1, 1, \sqrt{3})$ is on this sphere and add this point to your picture.

(b) Find a function $f(x, y)$ whose graph is the top-half of the sphere. Hint: solve for $z$.

(c) Imagine an ant walking along the surface of the sphere. It walks down the sphere along the path $C$ that passes through the point $P$ in the direction parallel to the $yz$-plane. Draw this path in your picture.

(d) Find a parametrization $r(t)$ of the ant’s path along the portion of the sphere shown in your picture. Specify the domain for $r$, i.e. the initial time when the ant is at $P$ and the final time when it hits the $xy$-plane.

4. As in 1(d), consider a reparametrization

$$f(s) = r(h(s))$$

of an arbitrary vector-valued function $r: \mathbb{R} \to \mathbb{R}^3$. Use the chain rule to calculate $|f'(s)|$ in terms of $r'$ and $h'$. 