Functions of several variables; Limits.

1. For each of the following functions \( f : \mathbb{R}^2 \to \mathbb{R} \), draw a sketch of the graph together with pictures of some level sets.

   (a) \( f(x, y) = xy \)

   (b) \( f(x) = |x| \). Please note here that \( x \) is a vector. In coordinates, this function is \( f(x, y) = \sqrt{x^2 + y^2} \).

   For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

2. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
   f(x, y) = \frac{2x^3 y}{x^6 + y^2} \quad \text{for} \ (x, y) \neq 0
\]

   In this problem, you'll consider \( \lim_{(x,y) \to 0} f(x, y) \).

   (a) Look at the values of \( f \) on the \( x \) - and \( y \) -axes. What do these values show the limit \( \lim_{(x,y) \to 0} f(x, y) \) must be if it exists?

   (b) Show that along each line in \( \mathbb{R}^2 \) through the origin, the limit of \( f \) exists and is 0.

   (c) Despite this, show that the limit \( \lim_{(x,y) \to 0} f(x, y) \) does not exist by finding a curve over which \( f \) takes on the constant value 1.

3. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
   f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for} \ (x, y) \neq 0
\]

   In this problem, you'll show \( \lim_{h \to 0} f(h) = 0 \).

   (a) For \( \epsilon = 1/2 \), find some \( \delta > 0 \) so that when \( 0 < |h| < \delta \) we have \( |f(h)| < \epsilon \). Hint: As with the example in class, the key is to relate \( |x| \) and \( |y| \) with \( |h| \).

   (b) Repeat with \( \epsilon = 1/10 \).

   (c) Now show that \( \lim_{h \to 0} f(h) = 0 \). That is, given an arbitrary \( \epsilon > 0 \), find a \( \delta > 0 \) so that that when \( 0 < |h| < \delta \) we have \( |f(h)| < \epsilon \).

   (d) Explain why the limit laws that you learned in class on Wednesday aren't enough to compute this particular limit.