1. Let $C$ denote the curve pictured at right, with the orientation shown.

   (a) For $\mathbf{F}(x, y) = (xy, 0)$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly. \textbf{(3 points)}

   Note that $\mathbf{F} = (0, 0)$ on $C_2$ and $C_3$, so
   
   $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} xy \, dx$

   $= \int_0^1 t(t-1) \, dt = \int_0^1 t^2 - t \, dt = \left[ t^3 - \frac{t^2}{2} \right]_{t=0}^{t=1} = -\frac{1}{6}$

   $x = t, \quad y = t-1 \ \Rightarrow \ \text{param of } C_1$

   $\int_C \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{6}$

   (b) Check your answer to part (a) using Green's Theorem. \textbf{(3 points)}

   $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_0^1 \int_{x-1}^1 -x \, dy \, dx$

   $= \int_0^1 \left. -x + x^2 \right|_0^1 \, dx = \left. -\frac{x^2}{2} + \frac{x^3}{3} \right|_0^1 = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$

2. For each function label its graph from among the options below: \textbf{(2 points each)}

   (A) $x^2 - y^2$ \quad (B) $\cos(xy)$ \quad (C) $e^{-(x^2+y^2)}$

   ![Graph Options]
3. (a) Each picture below depicts both (i) a constraint curve $C$ defined by $g(x, y) = 1$ for a function $g(x, y)$, and (ii) a level curve $f(x, y) = M$ of a function $f(x, y)$. Mark the boxes of all and only those pictures for which $M$ could be the maximum value of $f(x, y)$ subject to the constraint $g(x, y) = 1$. [In every picture, you should assume that $\nabla f$ is always nonzero.] (2 points)

(b) Suppose a function $f(x, y)$ attains its minimum value, subject to the constraint $2x^2 + 2xy^2 + y^3 = 5$, at $(x, y) = (1, 1)$. Assuming that $\nabla f(1, 1) \neq (0, 0)$, find a nonzero vector $v$ parallel to $\nabla f(1, 1)$. (3 points)

At a minimum, we will have the Lagrange condition:

$$\nabla f(1, 1) = \lambda \nabla g(1, 1) = \lambda \langle 6, 7 \rangle$$

$$\nabla g = \langle 4x + 2y^2, 4xy + 3y^2 \rangle$$

$$v = \langle 6, 7 \rangle$$

4. Suppose $f(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}$ has the table of values and partial derivatives shown at right. For $x(s, t) = s + 2t$ and $y(s, t) = s^2 - t$, let $F(s, t) = f(x(s, t), y(s, t))$ be their composition with $f$. Compute $\frac{\partial F}{\partial t}(2, 1)$. (3 points)

$$\frac{\partial F}{\partial t}(2, 1) = \frac{\partial f}{\partial x}(x(2, 1), y(2, 1)) \frac{\partial x}{\partial t}(2, 1) + \frac{\partial f}{\partial y}(x(2, 1), y(2, 1)) \frac{\partial y}{\partial t}(2, 1)$$

$$= 3 \cdot 2 + 1 \cdot (-1) = 5$$

$$\frac{\partial F}{\partial t}(2, 1) = 5$$
5. For each of the integrals

(A) \( \int_0^1 \int_0^1 \int_0^{1-x} f(x, y, z) \, dz \, dy \, dx \)  
(B) \( \int_0^1 \int_0^1 \int_0^y f(x, y, z) \, dx \, dy \, dz \)  
(C) \( \int_0^1 \int_x^{y-x} f(x, y, z) \, dz \, dy \, dx \)

label the solid corresponding to the region of integration below. (1 point each)

(A) \[ \text{ } \]  
(B) \[ \text{ } \]  
(C) \[ \text{ } \]

6. Compute the mass of solid region \( E \) shown at right if the mass density is \( \rho(x, y, z) = z \). (4 points)

\[
\int_0^1 \int_0^2 \int_0^x z \, dz \, dy \, dx = \int_0^1 \int_0^2 \left( \frac{z^2}{2} \right) \bigg|_0^x dy \, dx \\
= \int_0^1 \int_0^2 \frac{x^2}{2} \, dy \, dx = \int_0^1 X^2 \, dx = \left. \frac{X^3}{3} \right|_{X=0}^{X=1}
\]

\[
\text{Mass} = \frac{1}{3}
\]
7. (a) Let \( R \) be the region shown below right. Find a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) taking \( S = [0,1] \times [0,1] \) to \( R \). (3 points)

\[
T(u,v) = (au+bv, cu+dv)
\]

where

\[ T(1,0) = (a, c) = (1,1) \]

and

\[ T(0,1) = (b, d) = (-1, 2) \]

(b) Consider the transformation \( T(u,v) = (e^u - v, u + v) \) whose behavior is depicted below. Compute \( \iint_R 3 \, dA \) via an integral over \( S \). (3 points)

\[
J = \begin{pmatrix} e^u & -1 \\ 1 & 1 \end{pmatrix}
\]

\[ |\det J| = e^u + 1 \]

\[
\iint_R 3 \, dA = \iint_S 3 \cdot |\det J| \, du \, dv = 3 \int_0^1 \int_0^1 e^u + 1 \, du \, dv = 3 \int_0^1 e^u + u \bigg|_{u=0}^{u=1} \, dv = 3 \int_0^1 e^u \, dv
\]

\[ \iint_R 3 \, dA = 3e \]

8. Let \( S \) be the surface in \( \mathbb{R}^3 \) which is the boundary of the solid cube \( D = \{ -1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1 \} \). For \( F(x, y, z) = (yz^2 + x^2 + z, xe^z + x + y, xe^z + xy + z) \), compute \( \iint_S F \cdot n \, dS \) by any valid method, where \( n \) is the outward-pointing unit normal vector field. (4 points)

\[
\iiint_{\text{Cube}} d\mathbf{V} = \iiint_{\text{Cube}} 3 \, dV = 3 \, \text{Vol}(2^2) = 3 \cdot 2^3 = 24
\]

\[ \iint_S F \cdot n \, dS = 24 \]
9. Consider the region \( R \) below the surface \( z = 1 - x^2 - y^2 \) and above the \( xy \)-plane. Compute the volume of \( R \).
(5 points)

\[
\text{Volume} = \int_{0}^{2\pi} \int_{0}^{1} \int_{0}^{1-r^2} r \, dz \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{1} r (1-r^2) \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{1} r - r^3 \, dr \, d\theta = \int_{0}^{2\pi} \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=1} \, d\theta = \int_{0}^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}
\]

Volume = \( \frac{\pi}{2} \)

10. For each surface \( S \) in parts (a) and (b) give a parameterization \( \mathbf{r} : D \to S \). Be sure to explicitly specify the domain \( D \) and call your parameters \( u \) and \( v \).

(a) The portion of the surface \( x = y^2 \) shown at left. (2 points)

\[ \text{Take } u = y \text{ and } v = z \]

\[
D = \{ 0 \leq u \leq 1 \text{ and } 0 \leq v \leq 1 \}
\]

\[
\mathbf{r}(u,v) = \langle u^2, u, v \rangle
\]

(b) The portion of the cylinder \( x^2 + z^2 = 1 \) between the planes \( y = 0 \) and \( y = 2 \). (3 points)

\[ \text{params: } u = y \text{ (along cylinder, } v = \Theta, \text{ angle around } y \text{ axis) } \]

\[
D = \{ 0 \leq u \leq 2 \text{ and } 0 \leq v \leq 2\pi \}
\]

\[
\mathbf{r}(u,v) = \langle \cos v, u, \sin v \rangle
\]

(c) Let \( M \) be the surface in part (b). Is the surface integral \( \iint_{M} y \, dS \) negative, zero, or positive? Circle your answer. (1 point)


Circle your answer. (1 point)
11. Let $S$ be the surface parameterized by $\mathbf{r}(u, v) = \langle u, uv, v \rangle$ for $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$.

(a) Mark the picture of $S$ below. \(2\) points

(b) Completely setup, but do not evaluate, the surface integral $\iint_S x^2 \, dS$. \(5\) points

\[\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ 1 & v & 0 \\ 0 & u & 1 \end{vmatrix} = \langle v, -1, u \rangle\]

\[\iint_S x^2 \, dS = \iint_{-1}^{1} \iint_{-1}^{1} u^2 |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv\]

\[= \iint_{-1}^{1} \iint_{-1}^{1} u^2 \sqrt{1 + u^2 + v^2} \, du \, dv\]

\(-\text{cor to } u = 0 \text{ and } v = 0\)

(c) Find the tangent plane to $S$ at $(0,0,0)$. [You must show work that justifies your answer.] \(2\) points

Normal is $\mathbf{r}_u \times \mathbf{r}_v$ at $(0,0) = \langle 0, -1, 0 \rangle$

\[
\Rightarrow \text{plane is } y = 0
\]

Equation: $\begin{array}{c}
\square x+ \square y+ \square z= \square
\end{array}$
12. Consider the surface $S$ parameterized by $\mathbf{r}(u, v) = (\cos u, \sin u, v)$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

(a) Mark the picture of $S$ below. (2 points)

(b) Consider the vector field $\mathbf{F} = (yz, -xz, 1)$ which has $\text{curl} \mathbf{F} = (x, y, -2z)$. Directly evaluate $\iint_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, dS$ via the given parameterization, where $\mathbf{n}$ is the outward normal vector field. (4 points)

\[
\iint_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, dS = 2\pi
\]

(c) Check your answer in (b) using Stokes' Theorem. (4 points) Param for $C_1: \mathbf{r}(t) = (\cos t, \sin t, 0)$

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (0, 0, 1) \cdot (-\sin t, \cos t, 0) \, dt = \int_0^{2\pi} 0 \, dt = 0
\]

\[
\cos^2 t + \sin^2 t + 0 = 1
\]

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\cos t, -\sin t, 1) \cdot (\cos t, -\sin t, 0) \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.
\]

\[
\mathbf{r}_2(t) = (\sin t, \cos t, 1) \quad \text{flux} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0 + 2\pi = 2\pi
\]
13. Consider the function \( f(x, y) \) on the rectangle \( D = \{0 \leq x \leq 4 \text{ and } 0 \leq y \leq 2\} \) whose graph is shown below right. For each part, circle the best answer. \( (1 \text{ point each}) \)

(a) At the point \( P = (1, 0.5) \) is \( \frac{\partial f}{\partial y} \):

- negative
- zero
- positive

(b) At \( P \) is \( \frac{\partial^2 f}{\partial x^2} \):

- negative
- zero
- positive

(c) How many critical points does \( f \) have in the interior of \( D \)?

0 1 2 3 4

(d) The integral \( \iint_D f(x, y) \, dA \) is:

- negative
- zero
- positive

(e) For the curve \( C \) shown, the line integral \( \int_C \nabla f \cdot dr \) is:

\(-3 -1.5 0 1.5 3\)

(f) The line integral \( \int_C f \, ds \) is:

- negative
- zero
- positive

(g) Mark the plot of the vector field \( \nabla f \).
14. For each problem, circle the best answer. (1 point each)

(a) Consider the vector field \( \mathbf{F} = (1, x, -z) \). The vector field \( \mathbf{F} \) is: conservative not conservative

(b) Mark the plot of \( \mathbf{F} \) on the region where each of \( x, y, z \) is in \([0, 1]\):

(c) For the leftmost vector field in part (b) is the divergence: negative zero positive

Let \( S \) and \( H \) be the surfaces at right; the boundary of \( S \) is the unit circle in the \( xy \)-plane, and \( H \) has no boundary. Let \( \mathbf{G} = (x, y, z) \).

(d) The flux \( \iint_H \mathbf{G} \cdot \mathbf{m} \, dS \) is: negative zero positive

(e) The flux \( \iint_S \mathbf{G} \cdot \mathbf{n} \, dS \) is: negative zero positive

(f) The flux \( \iint_S (\text{curl} \mathbf{G}) \cdot \mathbf{n} \, dS \) is: negative zero positive