1. For the region $R$ at right, evaluate $\iint_R 2x\,dA$. (4 points)

\[
\iint_R 2x\,dA = \int_0^1 \int_{y}^{y=1} 2x\,dx\,dy
\]

\[
= \int_0^1 x^2 \bigg|_{x=0}^{x=1} dy = \int_0^1 y^2 - 1\,dy
\]

\[
= \frac{y^3}{3} - y \bigg|_{y=0}^{y=1} = \frac{1}{3} - 1 = -\frac{2}{3}
\]

\[
\iint_R 2x\,dA = -\frac{2}{3}
\]

2. Consider the ellipse $C$ given by $x^2 - xy + y^2 = 1$. Find all the points on $C$ which are closest to the origin. (6 points)

Want to minimize $f(x,y) = x^2 + y^2$ subject to

$g(x,y) = x^2 - xy + y^2 = 1$. Use Lagrange Multipliers.

$\nabla f = (2x, 2y) = \lambda \nabla g = \lambda (2x - y, 2y - x)$

\[
\Rightarrow 2x = \lambda (2x - y) \text{ and } 2y = \lambda (2y - x)
\]

\[
\Rightarrow \frac{1}{\lambda} = 1 - \frac{y}{2x} = 1 - \frac{x}{2y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2
\]

Case $x = y$: $g = x^2 = 1 \Rightarrow$ Two crit pts: $(1,1)$, $(-1,-1)$ where $f = 2$

Case $x = -y$: $g = 3x^2 = 1 \Rightarrow$ Two crit pts: $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, where $f = \frac{2}{3}$, so these are the closest pts

Closest points: $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
3. For each function: (a) $xy$ (b) $\cos(\sqrt{x^2 + y^2})$ (c) $e^x \cos y$

label its graph and its level set diagram from among the options below. Here each level set diagram consists of level sets $\{f(x, y) = c_i\}$ drawn for evenly spaced $c_i$. (9 points)
4. Let \( x(s, t) \) be the function whose contour plot is shown at right.

(a) Estimate \( \frac{\partial x}{\partial t}(2, 1) \) and circle the closest number below.

\[ \frac{\partial x}{\partial t}(2, 1) \approx \frac{-2}{1} = -2 \]

\[ \frac{\partial x}{\partial t}(2, 1): -6 -4 \boxed{-2} 0 2 4 6 \]

(b) Let \( g(x, y) \) be a function with the table of values and partial derivatives shown below and let \( y(s, t) = s + t \).

For \( G(s, t) = g(x(s, t), y(s, t)) \), compute \( \frac{\partial G}{\partial t}(2, 1) \).

\[ (x, y) \quad g(x, y) \quad \frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y} \]

\[ \begin{array}{c|c|c|c}
0, 3 & 0 & 3 & 6 \\
0, 1 & 2 & -3 & -1 \\
2, 1 & 3 & 4 & 7 \\
3, 3 & 1 & 3 & 5 \\
\end{array} \]

\[ g(x, y) = 0 \]
\[ y(2, 1) = 3 \]

\[ \frac{\partial G}{\partial t}(2, 1) = \frac{\partial g}{\partial x}(x(2, 1), y(2, 1)) \frac{\partial x}{\partial t}(2, 1) + \frac{\partial g}{\partial y}(x(2, 1), y(2, 1)) \frac{\partial y}{\partial t}(2, 1) = 1 \]

\[ = 3 \cdot (-2) + 6 \cdot 1 = 0 \]

\[ \frac{\partial G}{\partial t}(2, 1) = 0 \]

5. (a) For the curve \( C \) at right, directly compute \( \int_C y \, dx + 3x \, dy \). (5 points)

- **Bottom:** \( y = 0 \) and \( dy = 0 \) so \( \int_{C_1} = 0 \).
- **Right:** \( dx = 0 \) and \( x = 1 \), so \( \int_{C_2} = \int_0^1 3 \, dy = 3 \).
- **Diagonal:** \( r(t) = (1 - t, 1 - t) \) for \( 0 \leq t \leq 1 \). So \( dx = dy = -dt \)

and so \( \int_{C_3} y \, dx + 3x \, dy = \int_0^1 4(1 - t) \, dt = 2t^2 - 4 \bigg|_0^1 = -2 \)

\[ \int_C y \, dx + 3x \, dy = 1 \]

(b) Check your answer in (a) using Green's Theorem. (2 points)

\[ \int_C y \, dx + 3x \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_D 2 \, dA \]

\[ = 2 \text{ Area}(D) = 1 \]
6. Let $E$ be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$. Setup but do not evaluate a triple integral that computes the volume of $E$. (5 points)

$$\iiint_0^1 (1-x)(2-2x-2y) \, dz \, dy \, dx$$

![Tetrahedron diagram]

7. (a) Let $R$ be the region shown. Find a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [0,1] \times [0,1]$ to $R$. (3 points)

Use a linear transformation

$$T(u,v) = (au+bu, cu+dv)$$

Want:

$$T(1,0) = (a,b) = (2,1)$$

$$T(0,1) = (b,d) = (-1,1)$$

So $a=2$, $b=-1$, $c=1$, $d=1$

$$T(u,v) = (2u-v, u+v)$$

(b) Use your change of coordinates to evaluate $\iint_R x \, dA$ via an integral over $S$. (5 points)

**Emergency backup transformation:** If you can’t do (a), pretend you got the answer $T(u,v) = (uv,v)$ and do part (b) anyway.

$$\iint_R x \, dA = \iiint_0^1 (2u-v) 3 \, du \, dv = 3 \int_0^1 u^2 - uv \bigg|_0^1 \, dv$$

$$= 3 \int_0^1 (1-1) \, dv = 3 \int_0^1 (1-v^2) \bigg|_0^1 = \frac{3}{2}$$

$$\iint_S x \, dA = \frac{3}{2}$$
8. Let $R$ be the portion of the cylinder $x^2 + y^2 \leq 1$ which lies in the octant where \{ $x \geq 0, y \geq 0, z \geq 0$ \} and lies below the cone $z = 1 + \sqrt{x^2 + y^2}$. For the density $\rho = 6z$, compute the total mass of $R$. (7 points)

\[
\text{Mass} = \iiint_{R} \rho \, dV = \int_0^1 \int_0^1 \int_0^{\pi/2} 6z \cdot r \, d\theta \, dz \, dr \\
= \int_0^1 \int_0^{1+r} 3\pi z \cdot r \, dz \, dr \\
= \int_0^1 \frac{3\pi}{2} z^2 \Bigg|_{z=0}^{1+r} \, r \, dr \\
= \frac{3\pi}{2} \int_0^1 (r^2 + 2r + 1)r \, dr \\
= \frac{3\pi}{2} \left( \frac{r^3}{3} + \frac{2r^2}{2} + r \right) \Bigg|_0^1 \\
= \frac{3\pi}{2} \left( \frac{1}{3} + 1 + 1 \right) = \frac{17\pi}{2}.
\]

9. Let $\mathbf{F}(x, y, z) = \left( \frac{x^3}{3}, x^2 \cos(z) + \frac{y^3}{3}, \frac{z^3}{3} \right)$ and let $S$ denote the surface defined by $x^2 + y^2 + z^2 = 1$, equipped with the inward-pointing unit normal vector field $\mathbf{n}$. Compute $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ by any valid method. (6 points)

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = -\iiint_{\text{Ball}} \text{div} \mathbf{F} \, dV = -\iiint_{\text{Ball}} \frac{x^2 + y^2 + z^2}{\rho^2} \, dV
\]

\[
= -\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= -\int_0^{2\pi} \int_0^\pi \sin \phi \left[ \frac{1}{5} \rho^5 \right]_0^1 \, d\phi \, d\theta
\]

\[
= -\int_0^{2\pi} \left[ \frac{1}{5} \int_0^\pi \sin \phi \cos \phi \right]_0^\pi \, d\theta
\]

\[
= -\frac{2}{5} \left[ \int_0^{2\pi} \cos \phi \, d\theta \right] = -\frac{4\pi}{5}.
\]

\[
\iint_S \mathbf{F} \cdot \mathbf{n} \, dA = -\frac{4\pi}{5}.
\]
10. Consider the surface $S$ which is the portion of the plane $z = x + y$ which lies inside the cylinder $x^2 + z^2 = 4$. Give a parameterization $\mathbf{r}: D \to S$ where $D$ is a rectangle in plane with coordinates $u$ and $v$. (5 points)

Idea: Use rotated polar on cylinder:

$$\chi = v \cos u, \quad z = v \sin u$$

Now solve for $y$:

$$y = z - \chi = v \sin u - v \cos u$$

$$D = \{ 0 \leq u \leq 2\pi \text{ and } 0 \leq v \leq 2 \}$$

$$\mathbf{r}(u,v) = \langle v \cos u, v \sin u - \cos u, z \sin u \rangle$$

11. Consider the portion $S$ of the surface $z = 1 - x^2$ where $0 \leq y \leq 1$ and $z \geq 0$. Completely setup but do not evaluate $\iint_S x^2 \, dA$. (5 points)

Param: $\mathbf{r}(u,v) = (u,v,1-u^2)$

$$-1 \leq u \leq 1 \text{ and } 0 \leq v \leq 1$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & 0 \end{vmatrix} = \langle 2u, 0, 1 \rangle$$

$$\iint_S x^2 \, dA = \int_{-1}^1 \int_0^1 u^2 \| \mathbf{r}_u \times \mathbf{r}_v \| \, du \, dv$$

$$= \int_{-1}^1 \int_0^1 u^2 \sqrt{1 + 4u^2} \, du \, dv.$$
12. Let $S$ be the surface in $\mathbb{R}^3$ parameterized by $r(u, v) = (2 - 2v^2, v \cos u, v \sin u)$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

(a) Mark the correct picture of $S$ below. \hspace{1cm} (2 points)

(b) For the vector field $\mathbf{F} = (0, -z, y)$, directly evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA$ where $\mathbf{n}$ is unit normal vector field that points in the positive $x$-direction. \hspace{1cm} (5 points)

\[
\text{curl } \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & -z & y
\end{vmatrix} = (2, 0, 0)
\]

\[
\hat{\mathbf{n}} \times \hat{\mathbf{r}}_v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
0 & -v \sin u & v \cos u \\
-4v \cos u & \sin u & 0
\end{vmatrix} = (-v, 4v^2 \cos u, -4v^2 \sin u)
\]

\[
\iint_S (\text{curl } F) \cdot \hat{\mathbf{n}} \, dA = \int_0^{2\pi} \int_0^1 (2, 0, 0) \cdot (-v, 4v^2 \cos u, -4v^2 \sin u) \, dv \, du
\]

\[
= \int_0^{2\pi} \int_0^1 2v \, dv \, du = \int_0^{2\pi} v^2 \left|_{v=0}^{v=1} \right. \, du
\]

\[
= \int_0^{2\pi} 1 \, du = 2\pi
\]

\[
\iint_S (\text{curl } F) \cdot \mathbf{n} \, dA = 2\pi
\]

(c) Check your answer in (b) using Stokes' Theorem. \hspace{1cm} (3 points)

\[
\iint_S (\text{curl } F) \cdot \mathbf{n} \, dA = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(t) \cdot \mathbf{F}'(t) \, dt
\]

\[
\mathbf{F}(t) = (0, \cos t, \sin t) = \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \frac{2\pi}{2} = \pi
\]

\[
\mathbf{F}'(t) = (0, -\sin t, \cos t) = \int_0^{2\pi} 1 \, dt = \left[2\pi \right] \checkmark
\]
13. Consider the function \( f(x, y) \) on the rectangle \( D = \{0 \leq x \leq 3 \text{ and } 0 \leq y \leq 2\} \) whose contours are shown below right. For each part, circle the best answer. (1 point each)

(a) The maximum value of \( f \) on \( D \) is:

0 3 [6] 9 DNE

(b) At \( P \), the derivative \( \frac{\partial^2 f}{\partial y^2} \) is:

negative zero positive

(c) The value of \( D_{uf}(P) \) is:

negative zero positive

(d) The number of critical points of \( f \) in \( D \) which are saddles is:

0 1 2 3

(e) The integral \( \int_C f \ ds \) is:

negative zero positive

(f) The integral \( \int_C \nabla f \cdot dr \) is:

-4 -2 0 2 4

(g) The integral \( \iint_D f \ dA \) is:

-27 -18 -9 0 9 18 27

(h) Mark the plot below of the gradient vector field \( \nabla f \).
14. Consider the vector field $\mathbf{F}$ on $\mathbb{R}^2$ shown below right. For each part, circle the best answer. (1 point each)

(a) The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is:

- negative  
- zero  
- positive  

(b) At $A$, the vector curl$\mathbf{F}$ is:

- $(1, 0, 0)$  
- $(0, 0, -1)$  
- $(0, 0, 1)$  

(c) At $B$, the divergence div$\mathbf{F}$ is:

- negative  
- zero  
- positive  

(d) If $\mathbf{F} = \langle P, Q \rangle$, then $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \ dA$ is:

- negative  
- zero  
- positive  

(e) The vector field $\mathbf{F}$ is conservative:  

- True  
- False  

15. Consider the surfaces $S$ and $H$ show below right; the boundary of $S$ is the unit circle in the $xy$-plane, and $H$ has no boundary. For each part, circle the best answer.

(a) For $\mathbf{F} = \langle yz, xz + x, z \rangle$, the integral $\iint_H \mathbf{F} \cdot \mathbf{n} \ dA$ is:

- negative  
- zero  
- positive  

(b) The flux of curl$\mathbf{F} = \langle -x, y, 1 \rangle$ through $H$ is:

- negative  
- zero  
- positive  

(c) The integral $\iint_S (\text{curl}\mathbf{F}) \cdot d\mathbf{S}$ is:

- $-2\pi$  
- $-\pi$  
- $0$  
- $\pi$  
- $2\pi$  

(d) For $\mathbf{G} = \langle y, z, 2 \rangle$, the integral $\iint_S \mathbf{G} \cdot \mathbf{n} \ dA$ is:

- $-2\pi$  
- $-\pi$  
- $0$  
- $\pi$  
- $2\pi$  

SCRATCH WORK MAY GO HERE.