LECTURE 9

LAST TIME: PARTIAL DERIVATIVES \( \frac{df}{dx} = \text{slope of tangent line to slice of graph } y = \text{const}. \)

Easy to calculate: \( \frac{df}{dx}(x, y) = \sin(x)y + e^y \)
\( \frac{df}{dx} = \cos(x)y + e^y, \quad \frac{df}{dy} = \sin(x) + e^y \).

APPLICATIONS: RECALL O.D.E.'S = ORDINARY DIFFERENTIAL EQUATIONS

Eq \( \frac{dx}{dt} = cx \) — AN EQUATION INVOLVING FUNCTIONS AND DERIVATIVES. A SOLUTION IS A FUNCTION SO THAT THE EQUATION IS TRUE MODELS GROWTH/DECAY.

P.D.E.'S = PARTIAL DIFFERENTIAL EQUATIONS

Eq \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \) — HEAT EQUATION (TEMPERATURE)
\( u = u(x, y, z, t) \) IS SOLN DESCRIBES DISTRIBUTION OF HEAT OVER SPACE OVER TIME.

1-DIML CASE: \( \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x^2} \)

TEMP DECREASE

\[\begin{align*}
\text{TEMP INCREASE}
\end{align*}\]
TANGENT PLANES AND DIFFERENTIABILITY

\[ f : \mathbb{R}^2 \to \mathbb{R} \]

WHAT SHOULD TANGENT PLANE ABOVE \((a,b)\) BE?

GRAPH OF LINEAR APPROXIMATION

\[ L(x,y) = f(a,b) + \left( \frac{\partial f}{\partial x}(a,b) \right) (x-a) + \left( \frac{\partial f}{\partial y}(a,b) \right) (y-b) \]

To see this, fix \( x = a \) and vary \( y \), then fix \( y = b \) and vary \( x \).

\[ \text{[Compare } g : \mathbb{R} \to \mathbb{R} \text{ } \Rightarrow \text{ } \tilde{L}(x) = g(a) + g'(a)(x-a). \text{] } \]

SAY \( f \) is differentiable at \((a,b)\) if \( L \) approximates \( f \)

to first order at \((a,b)\):

\[ \lim_{(h,k) \to 0} \left| \frac{L(ath,btk) - f(ath,btk)}{|(h,k)|} \right| = 0 \]

\[ E(h,k) = f(ath,btk) - L(ath,btk) \]

\[ = \text{Error in linear approximation} \]

Ex.

\[ f(x,y) = 3 - x^2 - y^2 \quad (a,b) = (1,2) \]

\[ \frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -y \quad \Rightarrow \quad L(x,y) = \frac{5}{2} - (x-1) - (y-2) \]

\[ = \frac{7}{2} - x - y \]

\[ \text{Note: Just because } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ exist at } (a,b) \text{ does not mean } f \text{ is differentiable at } (a,b) \]
\textbf{Example} \quad f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}

\[
\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
\]

\[
\frac{\partial f}{\partial y}(0,0) = 0.
\]

But already seen \( \lim_{(x,y) \to (0,1)} f(x,y) = \text{DNE} \), so \( f \) is discontinuous at \( (0,0) \). This is a problem \( B/C \) of the \textbf{Theorem} \( f: \mathbb{R}^2 \to \mathbb{R} \). If \( f \) is differentiable at \( (a,b) \), then \( f \) is continuous at \( (a,b) \).

This is true because

\[
f(x,y) = f(a,b) + \left( \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \right) + \tilde{E}(x-a,y-b) \to 0
\]

as \( (x,y) \to (a,b) \).

So \( \lim_{(x,y) \to (a,b)} f(x,y) = f(a,b) \), and \( f \) is continuous at \( (a,b) \).

On the other hand, as soon as \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are nice enough, \( f \) is differentiable.

**Theorem** \( f: \mathbb{R}^2 \to \mathbb{R} \) if \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist and are continuous near \( (a,b) \), then \( f \) is differentiable at \( (a,b) \).

**Example** \( f(x,y) = 5 - x^2 - y^2 \) is differentiable since \( \frac{\partial f}{\partial x} = -2x, \frac{\partial f}{\partial y} = -2y \) are continuous on all \( \mathbb{R}^2 \).

- \( g(x,y) = xy e^x + \sin(xy) \) is differentiable since
  \[
  \frac{\partial g}{\partial x} = y e^x + xy e^x + y \cos(xy) \]
  \[
  \frac{\partial g}{\partial y} = 2xy e^x + x \cos(xy) \]
  Both \( \text{cont.} \).