LECTURE 6: §14.1, 12.6, 14.2. BRING LAPTOP TOMORROW IF YOU CAN

LAST TIME: STARTED THINKING ABOUT $f: \mathbb{R}^2 \to \mathbb{R}$

... GRAPHS & LEVEL SETS

$f(x, y) = y^2 - x^2$

$\n$

$f: \mathbb{R}^3 \to \mathbb{R} \quad \leftrightarrow \text{graph of } f = \{ (x, y, z, f(x, y, z)) \in \mathbb{R}^4 \}$

CAN'T DRAW THIS, BUT CAN STILL VISUALIZE LEVEL SETS.

EX: $f(x, y, z) = x^2 + y^2 + z^2$

LEVEL SETS ARE CONCENTRIC SPHERES

$x^2 + y^2 + z^2 = c$

EX: $g(x, y, z) = x + y + z$

LEVEL SETS ARE PARALLEL PLANES, ALL TO $c$.

$x + y + z = c$

EX: $F(x, y, z) = x^2 + y^2 - z^2$

Fix const $c$, look at

$x^2 + y^2 - z^2 = c$

TO SEE THIS LEVEL SET, INTERSECT IT WITH PLANES

$(x, 0, z) \cap x^2 - z^2 = c$

$xz$-plane

$(0, y, z) \cap y^2 - z^2 = c$

$yz$-plane
Intersections with horizontal planes are simplest:

\[ z = c' \]

\[ (x, y, c') \text{ with} \]

\[ x^2 + y^2 - (c')^2 = c \]

\[ x^2 y^2 = c(c')^2 \]

Symmetry: depends only on \( x^2 y^2 = c^2 \)

- Level sets invariant by rotations around \( z \)-axis

2-D analogue of conic sections called **quadric surfaces**

**Conic sections in \( \mathbb{R}^3 \):**

\[ A x^2 + B x y + C y^2 + D x + E y + F = 0 \]

- **Circle**
- **Ellipse**
- **Parabola**
- **Hyperbola**

**All are of this form (up to translating and rotating)**

**Or are degenerate**
**Quadric Surfaces in \( \mathbb{R}^3 \) (§ 12.4)**

\[ Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0 \]

**Ex. Ellipsoid**

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

**Elliptic paraboloid**

\[
\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}
\]

**Hyperbolic paraboloid**

\[
\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}
\]

double cone, hyperboloid of 1 or 2 sheets as above

\[
\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1
\]

*More in section tomorrow.*
LIMITS §14.2

To develop calculus for \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), we need limits...

Review limits for \( f : \mathbb{R} \rightarrow \mathbb{R} \).

\[
\lim_{x \to a} f(x) = L \text{ means what?}
\]

"\( f(x) \) is close to \( L \) provided \( x \) is sufficiently close to \( a \)."

WRITE * \( x = a + h \), so \( |h| = \) how close \( x \) is to \( a \)

* \( |x - a| = \) distance from \( x \) to \( a \).

* \( E(h) = f(a+h) - L = \) "Error" in evaluating \( f(a+h) \)

\[
|E(h)| = \text{size of error}
\]

So, \( \lim_{x \to a} f(x) = L \) also means:

"The Error \( |E(h)| \) is small, if \( h \) is sufficiently small".

Someone specifies ahead of time what "\( |E(h)| \) means for the error" — "Acceptable Error" — call it \( \varepsilon > 0 \). Need to be able to find "Tolerance" for \( h \) to achieve this level of accuracy — call this \( \delta > 0 \).

Given \( \varepsilon > 0 \) (read: acceptable error), there exist \( \delta > 0 \) (read: tolerance for \( h \)) so that

\[
|f(x) - L| < \varepsilon \text{ \ if } 0 < |h| < \delta.
\]
$\lim_{x \to 1} (2x+3) = 5$

$x = 1 + h$, $E(h) = 2(1+h)+3-5 = 2h$

**Acceptable Error $\varepsilon$**

If $0 < |h| < \frac{1}{10}$, then $|E(h)| = |2h| < 2 \cdot \frac{1}{10} = \frac{1}{5} = \varepsilon$.

**What if $\varepsilon = \frac{1}{100}$?**

$\varepsilon = \frac{1}{100}$

$\varepsilon = \frac{1}{1000}$

**Arbitrary $\varepsilon > 0$?**

$\varepsilon = \frac{1}{2}$, $\rightarrow$ Then if $0 < |h| < \delta$, not only choice — $\delta = \frac{1}{2}$ works, too! We have $|E(h)| = |2h| < 2\delta = 2\frac{1}{2} = 1$.

$\lim_{x \to 1} x^2-2x+1 = 0$

$x = 1 + h$, $E(h) = (1+h)^2 - 2(1+h) + 1 = 1 + 2h + h^2 - 2 - h + 1 = h^2$

$\varepsilon = \frac{1}{100} \rightarrow \delta = ?$... $\delta = \frac{1}{10}$, then $0 < |h| < \frac{1}{10}$ gives $|h|^2 = |(\frac{1}{10})|^2 = \frac{1}{100} = \varepsilon$

**Arbitrary $\varepsilon > 0$?**

$\varepsilon = \sqrt{\varepsilon} > 0$, $\rightarrow$ Then $0 < |h| < \delta$ gives $|h|^2 = |h|^2 < \delta^2 = \varepsilon$

... or $\frac{\sqrt{\varepsilon}}{100}$.