LECTURE 5: FUNCTIONS OF SEVERAL VARIABLES §14.

NOW THAT WE HAVE DEVELOPED SOME TOOLS FOR UNDERSTANDING & STUDYING $\mathbb{R}^n$, WE MOVE ON TO STUDYING FUNCTIONS WHOSE DOMAIN & RANGE ARE IN $\mathbb{R}^n$.

[BTW. MAKE SURE TO READ BACK OVER CHAPTER & NOTES FOR MATERIAL NOT DISCUSSED IN CLASS — e.g., TRIPLE PRODUCT]

EX. 1 A REAL VALUED FUNCTION OF A REAL VARIABLE: $f(x) = x^2 - 1$

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
</tr>
</tbody>
</table>

2 A REAL VALUED FUNCTION OF 2 REAL VARIABLES: $f(xy) = x^2 y^2$

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}$$

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$f(xy)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(2, -1)</td>
<td>5</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>13</td>
</tr>
</tbody>
</table>

3 AN $\mathbb{R}^2$-VALUED FUNCTION OF 3 REAL VARIABLES: $f(x, y, z) = (xy, yz)$

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

<table>
<thead>
<tr>
<th>$(x, y, z)$</th>
<th>$f(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1, 1)</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>(2, 0, 2)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>(1, 2, 3)</td>
<td>(-2, 6)</td>
</tr>
</tbody>
</table>

REAL WORLD EX. 1. TEMP IN ROOM $T : \mathbb{R}^3 \rightarrow \mathbb{R}$

2. NINJA DIRECTION & SPEED ON A MAP $W : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (FORCE FIELDS, ...)
Much of the general theory is already evident in functions $f: \mathbb{R}^2 \to \mathbb{R}$, and we start there.

**Graphs:** $f(x) = x^2 - 1 \implies$ graph of $f = \{ (x, f(x)) \in \mathbb{R}^2 \}$

What about $f(x,y) = x^2 + y^2$? $\implies$ graph of $f = \{ (x,y, f(x,y)) \in \mathbb{R}^3 \}$

One way to help visualize this is by intersecting with planes:

The graph of $f$ intersected with plane $y = 0$. 

\[(1, -2), (1, 1), (1.2, 5)\]
Intersecting with planes $z = c$ is particularly useful.

Essentially looking at
\[ \{ (x, y) \mid x^2 + y^2 = c^2 \} \]
for different values of $c$.

These are examples of level sets.

For a function $f : \mathbb{R}^n \to \mathbb{R}$, the level sets are sets:
\[ \{ (x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = c \} \]
for any constant $c$.

**Ex**
\[ f(x) = y^2 - x^2 \]

Try different values of $c$ to get different level sets: $c = -2, -1, 0, 1, 2$.
graph is a "saddle"
Example: Level sets of \( g(x,y) = x + 2y + 1 \)?

\[
x + 2y + 1 = c \\
\text{or} \\
x + 2y = (c - 1) \quad \text{constant} \\

c = 0 \quad \text{ALL LINES WITH SLOPE } -\frac{1}{2}
\]

**Example**

\[ r: \mathbb{R}^3 \rightarrow \mathbb{R} \]

\[ r(x,y,z) = x^2 + y^2 + z^2 \]

\[ \xi(x,y,z) \mid x^2 + y^2 + z^2 = c^2 \]

Level sets are concentric spheres.

**Example:** HOPE FIBRATION - SEE VIDEO

\( F: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) with range in unit sphere.