Lecture 21

Last time: Two theorems on conservative vector field, i.e., those \( \mathbf{F} \) with \( \mathbf{F} = \nabla f \).

**Theorem** A \( \mathbf{F} : D \rightarrow \mathbb{R}^n \) a continuous vector field on connected open set \( D \) (e.g., all of \( \mathbb{R}^n \)), then \( \mathbf{F} \) is conservative if and only if all line integrals are path independent.

*Already know:* \( \mathbf{F} \) conservative \( \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} \) path indep.

So, suppose \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is path indep.

How do we find a potential function \( f \) for \( \mathbf{F} \)?

Pick some \( A \), define
\[
\tilde{f}(x) = \int_{C_x} \mathbf{F} \cdot d\mathbf{r}
\]
where \( C_x \) is a path from \( A \) to \( x \).

(Any path!)

Why is \( \nabla \tilde{f} = \mathbf{F} \)?
So, if path independent, there exists a potential function, how do we find it?
\[ F(x, y) = (2x \cos y + e^x, -x^2 \sin y + 2y) \]

This is conservative... potential function \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \)

with \( f_x = 2x \cos y + e^x, f_y = -x^2 \sin y + 2y \)

\( f(x, y) \) is an antiderivative w.r.t. \( x \) of \( 2x \cos y + e^x \)

and \( " " " " " " " " " " " " y \) of \( -x^2 \sin y + 2y \)

\[ \Rightarrow f(x, y) = x^2 \cos y + e^x + g(y) \leftarrow \text{constant w.r.t. } x \]

\[ f(x, y) = x^2 \cos y + y^2 + h(x) \leftarrow \text{constant w.r.t. } y. \]

\[ \Rightarrow e^x + g(y) = y^2 + h(x) \]

A potential function \( f(x, y) = x^2 \cos y + e^x + y^2 \)

\[ \nabla \]

2nd Thm:

**Theorem 2**: \( F = P \hat{i} + Q \hat{j}: \mathbb{D} \rightarrow \mathbb{R}^2 \) continuously differentiable vector field, \( \mathbb{D} \) open, simply connected set. Then \( F \) is conservative if and only if \( P_y = Q_x \).

**Ex**: \( \mathbb{D} = \{(x, y) | (x, y) \neq (0, 0) \} \)

\[ F: \mathbb{D} \rightarrow \mathbb{R}^2 \) given by

\[ F = P \hat{i} + Q \hat{j} = \frac{-y}{x^2 + y^2} \hat{i} + \frac{x}{x^2 + y^2} \hat{j} \Rightarrow P_y = Q_x \]

C unit circle => \( \int_C F \cdot dr = 2\pi \)
JUST LOOK AT $\vec{F}$ ON A SIMPLY CONNECTED SET

$E \subseteq \{(x, y) | x > 0\} = D_0$

$\vec{F} = \nabla f$

$w/ \ f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

(check it!). \textbf{OBS:} $f(x, y) = \text{ANGLE w/ POS. X-AXIS}$

"ANGLE FUNCTION" DOES NOT EXTEND TO ALL OF D.

\textbf{NOTE:} ON $D_0$ ABOVE ANY TWO CHOICES $f$ ANGLE FUNCTION DIFFER BY A CONSTANT (MULTIPLE OF $2\pi$).

DERIVATIVE OF ANGLE FUNCTION

\textbf{DOES MAKE SENSE} — THIS IS $\vec{F}$.