LECTURE 20

LAST TIME: FUND. THM OF LINE INTEGRALS

\[ \int_C \nabla f \cdot ds = f(B) - f(A) \]

EX: CALCULATE THE WORK DONE BY \( F = \frac{1}{(x^2+y^2)^3} (xi+yi) \)

ACTING ON A PARTICLE TRAVERSING THE LINE SEGMENT \( C \)

FROM \((1,2)\) TO \((3,1)\).

\[ \text{AROM: } \overrightarrow{F}(t) = (1,2) + t(3,1) - (1,2) \]
\[ = (1+2t, 2-t) \]

CALCULATE:

\[ \int_C \overrightarrow{F} \cdot ds = \int_0^1 \frac{(1+2t, 2-t)}{(1+2t)^2 + (2-t)^2} \cdot (2, -1) \, dt \]

\[ = \int_0^1 \frac{2+4t - (2-t)}{(1+4t+4t^2 + 4-4t + t^2)^2} \, dt \]

\[ = \int_0^1 \frac{5t}{(5+5t^2)^2} \, dt = \frac{1}{2} \int_0^1 \frac{du}{u^{\frac{1}{2}}} = -u^{\frac{1}{2}} \bigg|_0^1 \]

\[ u = 5+5t^2 \]
\[ du = 10t \, dt \]

\[ = \frac{-1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \frac{12 - 1}{15} = \frac{11}{15} \]
This is an example of a gravity-like field, so recall for \( F(x, y) = \frac{-1}{\sqrt{x^2 + y^2}} \Rightarrow \nabla f = F \), and by FTC: \[
\int_C F \cdot \, d\mathbf{r} = f(C(1)) - f(C(0)) = -\frac{1}{\sqrt{1+1}} - \frac{-1}{\sqrt{1+2}} = \frac{\sqrt{2} - 1}{10}.
\]

Recall \( \bar{F} : \mathbb{R}^n \to \mathbb{R}^n \) is conservative \( \iff \bar{F} = \nabla f \) for some \( f : \mathbb{R}^n \to \mathbb{R} \).

\[\Rightarrow\] Line integrals \( \int_C F \cdot \, d\mathbf{r} \) are path independent.

Path independence:

1. \( \int_{C_1} \bar{F} \cdot \, d\mathbf{r} = \int_{C_2} \bar{F} \cdot \, d\mathbf{r} \) when \( C_1, C_2 \) connect the same two points:

\[\text{or}\]

2. \( \int_C \bar{F} \cdot \, d\mathbf{r} = 0 \) for every closed curve \( C \) (starts and ends at same point).

Why same?

\( C = (C_1) \cup (-C_2) \)

\[0 = \int_C \bar{F} \cdot \, d\mathbf{r} = \int_{C_1} \bar{F} \cdot \, d\mathbf{r} + \int_{-C_2} \bar{F} \cdot \, d\mathbf{r} = \int_{C_1} \bar{F} \cdot \, d\mathbf{r} - \int_{C_2} \bar{F} \cdot \, d\mathbf{r} \]
**NON CONSERVATIVE VECTOR FIELDS**

**EX** \[ F = (-y, x) = -y \hat{i} + x \hat{j} \] (ROTATION VECTOR FIELD)

\[ \int_C F \cdot dr = 2\pi \quad \text{C CLOSED CURVE = UNIT CIRCLE (CALC BEFORE)} \]

**EX** \[ F = (xy^2, xy) = xy^2 \hat{i} + xy \hat{j} \] CONSERVATIVE?

TRY UNIT CIRCLE : C  
\[ \int_C F \cdot dr = \int_C (\cos t \sin^3 t, \cos t \sin t) \cdot (-\sin t, \cos t) \, dt \]
\[ = \int_0^{2\pi} -\sin^4 t \cos t + \sin t \cos^3 t \, dt = 0 \quad \text{EASY CALC.} \]

TRY SOMETHING ELSE?

\[ \int_C F \cdot dr = \int_0^1 (t^7, t^3) \cdot (1, 2t) \, dt = \int_0^1 t^7 + 2t^4 \, dt \]
\[ = \frac{t^8}{8} + \frac{2t^5}{5} \bigg|_0^1 = \frac{1}{8} + \frac{2}{5} = \frac{19}{40} \]

\[ C_1 : F_1 : (t, t^3) \]
\[ C_2 : F_2 : (t+1, t+1) \]

\[ \int_{C_1} F \cdot dr = \int_0^1 (t^7, t^3) \cdot (1, 1) \, dt = \int_0^1 t^7 + t^6 \, dt \]
\[ = \frac{t^8}{8} + \frac{t^5}{5} \bigg|_0^1 = \frac{1}{8} + \frac{1}{5} = \frac{13}{40} \neq \frac{19}{40} \]

NOT PATH DEPENDENT!

\[ \Rightarrow \] NOT CONSERVATIVE.

**HOW CAN WE TELL?**

**SUPPOSE** \[ \nabla f = F = (P, Q) \Rightarrow P = f_x, Q = f_y \]

So \[ P_y = f_{xy} = f_{yx} = Q_x \quad \text{i.e.} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \]
Look at Ex. above again:

\[ P = x y^3, \quad Q = x y \Rightarrow P_y = 3 x y^2 \quad Q_x = y \neq P_y \Rightarrow \text{not cons} \]

\[ \text{Ex. } \quad \vec{F} = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) : D \to \mathbb{R}^2 \quad D = \{ (x, y) \mid (x, y) \neq (0, 0) \} \]

Not conservative,

\[ \forall \Gamma \quad \frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) = \frac{-(x^2 y^3) + y(2y)}{(x^2 + y^2)^2} \]

\[ = \frac{y^2 - x^2}{(x^2 + y^2)^2} \]

\[ \frac{\partial}{\partial x} \left( \frac{x}{x^2 + y^2} \right) = \frac{x^2 y^2 - x(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial y} \left( \frac{-y}{x^2 + y^2} \right) \]

So, \( P_y = Q_x \) isn't enough... what else? — depends on \( D \)

A subset \( D \subset \mathbb{R}^2 \):

\( D \) is open if every point has a disk with that center contained in \( D \)

\[ \text{Ex. } D = \{ (x, y) \mid 0 < x < 1, 0 < y < 1 \} \]

\[ \text{Not Ex. } D = \{ (x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \} \]

\( D \) is open if the complement (points not in \( D \)) is closed
\[ D \text{ is connected if any two points are joined by a path in:} \]

\[ \text{EX} \quad \text{NON-EX} \]

\[ D \text{ is simply connected if connected \& no "holes"} \]

\[ \text{EX} \quad \text{NON-EX} \]

**Theorem A** 
\[ \vec{F}: D \rightarrow \mathbb{R}^n \]  
A vector field on a connected open set, then \( \vec{F} \) is conservative if and only if \( \int_C \vec{F} \cdot d\vec{r} \) is path independent.

**Theorem B** 
\[ \vec{F} = (p, q): D \rightarrow \mathbb{R}^2 \]  
A vector field on an open simply connected set \( D \), then \( \vec{F} \) is conservative if and only if \( \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \) on \( D \)

In previous example, \( \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \) on \( D = \{(x,y) | (x,y) \neq (0,0)\} \),

\[ \text{but \ D \ is \ not \ simply \ connected} \]

\[ \text{want to define} \quad f(x,y) = \text{"angle" in } \vec{F} \text{ makes with the x-axis"} \]

\[ f = \arctan \left( \frac{y}{x} \right) \]

\[ \text{check } \nabla f = \vec{F} \]

\[ \text{but the angle isn't well defined} \]