Lecture 16  Space Curves §13.1–13.3 (Intro)

Parameterized curve in \( \mathbb{R}^2 \): \( \mathbf{x} = x(t), y = y(t) \)

--- In \( \mathbb{R}^3 \): \( \mathbf{x} = x(t), y = y(t), z = z(t) \).

Can also write this as a vector-valued function

\[ \mathbf{r} : \mathbb{R} \to \mathbb{R}^2 \text{ or } \mathbb{R}^3 \]

\[ \mathbf{r}(t) = (x(t), y(t)) \quad \text{or} \quad \mathbf{r}(t) = (x(t), y(t), z(t)) \]

**Ex Circle**

Component Functions

\[ \mathbf{r}(t) = (\cos(t), \sin(t)) \]

Clockwise

\[ \mathbf{r}(t) = (\cos(\pi - t), \sin(\pi - t)) \]

\[ = (\cos(t), -\sin(t)) \]

Clockwise

--- Not just circle, also have info about how we traverse it...

**Ex Helix**

\[ \mathbf{r}(t) = (\cos(t), \sin(t), t) \]

"Spring shape"

\[ \mathbf{r}(t) = (\cos(t), \sin(t), \frac{t}{10}) \]

**Ex Lines**

"Point" "direction" \[ \mathbf{r}(t) = \mathbf{a} + t \mathbf{b} \]

\[ \mathbf{r}(t) = (1, -1) + t (1, 3) = (1 + t, -1 + 3t) \]

fixed vectors
PARAMETRIZE / FIND VECTOR FUNCTION FOR INTERSECTION:

\[ z = y^2 - x^2 \quad \text{AND} \quad x^2 + y^2 = 1 \]

Note that \( x, y \) coordinates constrained to \( x^2 + y^2 = 1 \) and \( z \)-coordinate is a function of \( x, y \) coordinates:

\[
\vec{f}(t) = (\cos(t), \sin(t), \sin^2(t) - \cos^2(t))
\]

\[
= (\cos(t), \sin(t), -\cos(2t))
\]

Why do this? It gives more info by different perspective.

[See book for more examples and visualization exercises/exercises.]

LIMITS:

\[
\lim_{t \to a} \vec{f}(t) = \vec{0}
\]

Means \( \| \vec{f}(t) - \vec{0} \| \) is small when \( |t-a| \) is small.

FACT: If \( \vec{f}(t) = (x(t), y(t), z(t)) \), then

\[
\lim_{t \to a} \vec{f}(t) = \left( \lim_{t \to a} x(t), \lim_{t \to a} y(t), \lim_{t \to a} z(t) \right)
A function \( f \) is continuous at \( a \) if \( \lim_{t \to a} f(t) = f(a) \).

---

Just continuity of \( x(t), y(t), \) and \( z(t) \) at \( a \).

Derivative? - Already mentioned this earlier.

\[
\mathbf{F}'(t) = (x'(t), y'(t), z'(t)) = \lim_{h \to 0} \frac{\mathbf{F}(t+h) - \mathbf{F}(t)}{h}
\]

Why? - Fact above.

Acceleration? - How velocity changes.

\[
\mathbf{F}''(t) = (x''(t), y''(t), z''(t)) = ?
\]

A vector, \( \mathbf{F}'(t) \) is a position of particle at time \( t \).

Distance:

\[
\|\mathbf{F}'(t)\| = \lim_{h \to 0} \frac{\|\mathbf{F}(t+h) - \mathbf{F}(t)\|}{h}
\]

\[
\|\mathbf{F}'(t)\| \to \text{time}
\]

\[
\|\mathbf{F}'(t)\| \to \text{time to disp.}
\]

Limit is velocity vector (think \( \mathbf{F}'(t) \) is position of particle at time \( t \)).

Example:

\[
\mathbf{F}'(t) = (\cos t, \sin t)
\]

\[
\mathbf{F}''(t) = (-\sin t, \cos t)
\]

\[
\mathbf{F}'''(t) = (-\cos t, -\sin t) = -\mathbf{F}'(t)
\]

Constant speed but not constant velocity — direction of velocity is changing.
Length of a curve: \( \mathbf{F}: [a,b] \to \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

At constant speed, length is speed \times time.

Not necessarily constant:

\[
\mathbf{F}(t) \approx \mathbf{F}(t') \quad \text{for any } t \in [t_i, t_{i+1}]
\]

Length \( \approx \sum_{i=1}^{n} |\mathbf{F}'(t_i)| (t_i - t_{i-1}) \rightarrow \int_{a}^{b} ||\mathbf{F}'(t)|| \, dt \)

Length = \( \int_{a}^{b} ||\mathbf{F}'(t)|| \, dt \)

Alternative perspective

Approximate by straight segments

Length \( \approx \sum_{i=1}^{n} |\mathbf{F}(t_i) - \mathbf{F}(t_{i-1})| \approx \sum_{i=1}^{n} |\mathbf{F}'(t_i)| (t_i - t_{i-1}) \rightarrow \int_{a}^{b} |\mathbf{F}'(t)| \, dt \)

Example: Helix (a piece of it):

\[
\mathbf{F}(t) = (\cos t, \sin t, t) \quad t \in [0, 4\pi]
\]

Length \( \approx \int_{0}^{4\pi} |\mathbf{F}'(t)| \, dt = \int_{0}^{4\pi} \sqrt{1 + \sin^2 t + \cos^2 t} \, dt = 9\sqrt{2} \)