LECTURE 10  
CHAIN RULE §14.5

LAST TIME: \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) is DIFFERENTIABLE AT \((a,b)\) IF

\[
f(a+h,b+k) = f(a,b) + (\frac{\partial f}{\partial x}(a,b))h + (\frac{\partial f}{\partial y}(a,b))k + E(h,k) = L(a+h,b+k) + E(h,k) \]

AND

\[
\lim_{(h,k) \to (0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0 \quad \text{Z=L(x,y) IS EVEN FOR TANGENT PLANE.}
\]

THEOREM 1: IF \( \frac{\partial f}{\partial x} \) AND \( \frac{\partial f}{\partial y} \) EXIST AND ARE CONTINUOUS NEAR \((a,b)\), THEN \( f \) IS DIFFERENTIABLE AT \((a,b)\).

EX: \( f(x,y) = 3-x^2-y^2 \) \( \frac{\partial f}{\partial x} = -2x \) \( \frac{\partial f}{\partial y} = -2y \) CONT.

SO, \( f \) IS DIFFERENTIABLE EVERYWHERE.

INSTEAD OF \( h,k \) WE OFTEN WRITE

\[
\Delta x = h = x-a \quad \Delta y = k = y-b
\]

THEN WE HAVE

\[
\Delta f = f(x,y) - f(a,b) \approx f_x(a,b) \Delta x + f_y(a,b) \Delta y
\]

=gapprox.

Therefore, you all know: \( f,g: \mathbb{R} \rightarrow \mathbb{R} \), THEN

\[
(f \circ g)'(t) = f'(g(t)) g'(t) \quad \text{CHAIN RULE}
\]

EX: \( h(t) = \cos(t^3) = -3 \sin(t^3) 2t \)
Now suppose we have

1. \( f : \mathbb{R}^2 \to \mathbb{R} \) (and)

2. A parametrized curve \((x(t), y(t))\), \(x, y : \mathbb{R} \to \mathbb{R}\)
   or \((x, y) : \mathbb{R} \to \mathbb{R}^2\)

We can compose these:

\[ h(t) = f(x(t), y(t)) \] so \( h : \mathbb{R} \to \mathbb{R} \)

What is \( h'(t) \)?

\[ h(t+\Delta t) \approx h(t) + h'(t) \Delta t \]

Let's find \( h'(t) \)

\[
\begin{align*}
X(t+\Delta t) &= X(t) + x'(t) \Delta t + E_x(\Delta t) \\
y(t+\Delta t) &= y(t) + y'(t) \Delta t + E_y(\Delta t)
\end{align*}
\]

\[
f(x+\Delta x, y+\Delta y) \approx f(x, y) + f_x(x, y) \Delta x + f_y(x, y) \Delta y
\]

\[ h(t+\Delta t) = f(x(t+\Delta t), y(t+\Delta t)) \]

\[
= f \left( \frac{x(t) + x'(t) \Delta t + E_x(\Delta t)}{x}, \frac{y(t) + y'(t) \Delta t + E_y(\Delta t)}{y} \right)
\]

\[
\approx f(x(t), y(t)) + f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t) \Delta t
\]

\[
\Rightarrow h'(t) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)
\]

\[ A1 \]
ALT. NOTATION IS USEFUL TO REMEMBER THIS
\[ z = f(x, y), \quad x = x(t), \quad y = y(t) \]
\[ z(t) = f(x(t), y(t)) \quad \text{and} \]
\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad \text{"CANCEL"}
\]
Example
\[ f(x, y) = (x^2 + 2xy + y)^3 \quad x = \cos(t), \quad y = \sin(t) \]
\[ h(t) = f(x(t), y(t)) \]
\[ h'(t) = ? ? \]
\[ f_x = 3(x^2 + 2xy + y)^2 (2x + 2y) \]
\[ f_y = 3(x^2 + 2xy + y)^2 (2x + 1) \]
\[ x' = -\sin t \]
\[ y' = \cos t \]
So
\[ h'(t) = 3(\cos^2 t + 2 \cos t \sin t + \sin t)^2 (2 \cos t + 2 \sin t) (-\sin t) \]
\[ + 3(\cos^2 t + 2 \cos t \sin t + \sin t)^2 (2 \cos t + 1) (\cos t) \]

CHAIN RULE: VALID IN ANY DIM: \( f: \mathbb{R}^n \rightarrow \mathbb{R} \)
Example
\[ u = f(x, y, z, w) = x^2 + yzw \quad x = t, \quad y = t^2, \quad z = t^3, \quad w = t^4 \]
\[ \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} + \frac{\partial u}{\partial w} \frac{dw}{dt} = 2x(t)(1) + 2y(t)w(t)(2t) + y(t)w(t)(3t^2) \]
\[ = 2t + t^7 (2t) + t^6 (3t^2) + t^5 (4t^3) \quad \text{CAN CHECK:} \quad u = t^2 + t^9 + t^{15} + t^{16} \]
\[ \frac{du}{dt} = 2t + 9t^8 \]