Midterm Exam 3 Solutions, Comments, and Feedback

**Grading:** The raw scores were adjusted by adding 10 points to each score. With this adjustment, the median and average came out to be 79/90, and the highest scores 98/90 (2), 89/90, 87/90. **If you have any questions about the grading, just ask. I’d be happy to discuss your work with you.**

As usual, you can access all your scores, and your current score total, via a link on the course webpage. If there is an error in the score display (e.g., missing or incorrect score), let me know. **Note that the scores shown online are the adjusted scores, not the raw scores.**

**Solutions:** Solutions, along with some remarks about common errors, are attached. **Check the solutions and the comments below first before asking questions about the grading.**

**Order of quantifiers in Cauchy Criterion (Problem 4(d) and Problem 3).** As we have seen in connection with formal logical statements, changing the order of quantifiers in a logical statement can completely change the meaning of the statement; see Problem 2.24 from HW 2, and Problem 6 in Honors HW 1 for some drastic illustrations of this point.

The Cauchy Criterion is another example that illustrates the importance of the order of the quantifiers. Getting this right was crucial for a correct analysis of Problem 4(d), which asked if \( \lim_{n \to \infty} (a_n + k - a_n) = 0 \) for all \( k \in \mathbb{N} \) implies \( \lim_{n \to \infty} a_n \) exists. As was discussed in class, there is a key difference between a condition of the type \((*)\) and the Cauchy criterion, and the two conditions are not equivalent. This can be seen as follows: Cauchy’s Criterion states

\[
(1) \quad (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n, m \geq N)[|a_n - a_m| < \varepsilon]
\]

By symmetry, we may assume \( m > n \) here, so the quantifier \( \forall n, m \geq N \) (which means the same as \( (\forall n \geq N)(\forall m \geq N) \)) can be replaced by \( (\forall n \geq N)(\forall n > m) \), without changing the meaning of the statement. Moreover, making the change of variables \( m = n + k \), the latter condition can be written as \( (\forall m \geq N)(\forall k \geq 1) \) or \( (\forall m \geq N)(\forall k \in \mathbb{N}) \). Thus, (1) is equivalent to

\[
(2) \quad (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(\forall k \in \mathbb{N})[|a_{n+k} - a_n| < \varepsilon]
\]

On the other hand, using the \( \varepsilon \)-definition of a limit, condition \((*)\), i.e., \( \lim_{n \to \infty} (a_{n+k} - a_n) = 0 \) for all \( k \in \mathbb{N} \), becomes

\[
(3) \quad (\forall k \in \mathbb{N})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_{n+k} - a_n| < \varepsilon]
\]

The only difference between (2) and (3) is in the placement of the quantifier \( \forall k \in \mathbb{N} \): In (3) this quantifier occurs at the outermost level, whereas in (2) it occurs at the innermost level. This makes a crucial difference: In (3) \( N \) is chosen after \( k \), so it can be “customized” to the given \( k \)-value. By contrast, in (2), the same \( N \) has to “work” for all \( k \). As a result, (2) (i.e., Cauchy’s Criterion) is a stronger condition than (3). Any sequence satisfying (2) also satisfies (3), but the converse is not true: the sequence \( a_n = \sqrt{n} \) satisfies (3), but it is unbounded, hence cannot satisfy the Cauchy Criterion.

** (A) \( \lim a_n \) and (B) \( \lim b_n \), versus (C) \( \lim a_n b_n \). (Problem 4(b)) This problem dealt with the relation between existence of these limits. In one direction, the situation is very simple: If the limits (A) and (B) exist, then so does the limit (C), by a direct application of the product formula for limits.

The problem was about a different, and a bit trickier, situation: Does the existence of limit (A), along with the assumption that this limit equals 1, and the nonexistence of limit (B) imply the nonexistence of limit (C)? The answer turns out to be yes, but requires a careful argument, via contradiction: Assume limits (A) and (C) exist, but (B) does not exist, and also that the limit (A) is equal to 1. Then write \( b_n = (a_n b_n) (1/a_n) \), use the property that \( \lim a_n = 1 \) implies \( \lim 1/a_n = 1 \) (proved in the homework), along with the product formula for limits, to conclude that \( \lim b_n \) exists and equals \( \lim (a_n b_n) \lim (1/a_n) = \lim (a_n b_n) \), contradicting the assumption.

It would be wrong to start out writing \( \lim (a_n b_n) = \lim a_n \lim b_n \), as this product formula is valid only when both limits on the right are known to exist (which is not the case for the second limit).

****
1. For the following questions an answer is sufficient: Just state the requested definition or theorem.

(a) State the Bolzano-Weierstrass Theorem.

**Solution:** Any bounded sequence contains a convergent subsequence.

(b) State the Completeness Axiom.

**Solution:** Any nonempty set of real numbers that has an upper bound has a least upper bound.

(c) Give a formal definition of the “limsup” of a sequence \( \{a_n\}_{n=1}^{\infty} \). (Be sure to use correct mathematical notation. Your definition may involve the “sup” notation.)

**Solution:** \( \limsup_{n \to \infty} a_n = \lim_{n \to \infty} \sup \{a_k : k \geq n\} \).

(d) Give a formal definition of a partition of a set \( S \). (Be sure to use proper set-theoretic terminology and notation.)

**Solution:** A partition of \( S \) is a collection of subsets \( A_i \) of \( S \) that are (i) nonempty, (ii) pairwise disjoint (i.e., \( A_i \cap A_j = \emptyset \) for all \( i \neq j \)), and (iii) their union is the set \( S \) (i.e., \( \bigcup_i A_i = S \)).

2. Consider the relation on \( \mathbb{R} \) defined by

\[ x \sim y \iff \text{there exists } q \in \mathbb{Q} - \{0\} \text{ such that } x = qy. \]

(a) Prove that this relation is an equivalence relation.

(b) Determine the equivalence class of 347 for this relation (i.e., give an explicit description of all the elements in this class).

(c) Show that the relation defined by \( x \sim y \iff \text{there exists } q \in \mathbb{Q} \text{ such that } x = qy \) (i.e., with \( \mathbb{Q} \) in place of \( \mathbb{Q} - \{0\} \)) is not an equivalence relation.

**Solution:** (a) To show that the relation is an equivalence relation, we need to show that it is reflexive, symmetric, and transitive.

**Reflexive:** For any \( x \in \mathbb{R} \) we have \( x = 1 \cdot x \), and since \( 1 \in \mathbb{Q} \), this proves \( x \sim x \).

**Symmetric:** Suppose \( x, y \in \mathbb{R} \) and \( x \sim y \). Then there exists \( q \in \mathbb{Q} - \{0\} \) such that \( x = qy \).

Since \( q \neq 0 \), we can divide by \( q \), getting \( y = (1/q)x \). Since \( q \) is rational and non-zero, so is \( 1/q \). Hence \( y \sim x \).

**Transitive:** Let \( x, y, z \in \mathbb{R} \) and suppose \( x \sim y \) and \( y \sim z \). Then there exist \( q, r \in \mathbb{Q} - \{0\} \) such that \( x = qy \) and \( y = rz \). Therefore \( x = q(rz) = (qr)z \). Since \( q \) and \( r \) are rational and both non-zero, \( qr \) is also rational and non-zero. Therefore \( x \sim z \).

(b) By definition, the equivalence class of 347 is the set of the real numbers \( x \) satisfying \( x \sim 347 \), i.e., the set of all real numbers that can be written in the form \((*) x = q \cdot 347 \) for some nonzero rational number \( q \). To get a simple explicit description of this set, note that (i) every number of the form \((*) \) is clearly rational and nonzero, and (ii) every nonzero rational number \( x \) can be expressed in the form \((*) \) (by taking \( q = x/347 \)). Hence the equivalence class of 347 is simply \( \mathbb{Q} - \{0\} \), i.e., the set of all nonzero rational numbers.

(c) With \( \mathbb{Q} \) in place of \( \mathbb{Q} - \{0\} \), the relation is reflexive, and transitive, but the symmetry property does not hold. For example, for \( x = 1, y = 0 \). we have \( x \neq qy \) for any \( q \), so \( x \not\sim y \), but \( y \sim 0x \), and since \( 0 \in \mathbb{Q} \), this shows \( y \sim x \).

3. (a) Using only the definitions of “Cauchy sequence” and “bounded sequence”, prove that any Cauchy sequence is bounded. Be sure to include any necessary quantifiers, in the correct order. (The proof should be done directly from these definitions and not use any of the properties, theorems, propositions, etc. established in the book, in class, or in the hw problems.)
4. For each of the following statements, determine if it is true or false. If it is true, give a brief justification (e.g., by citing an appropriate theorem or property); if it is false, give a specific counterexample.

(a) If \( \lim_{n \to \infty} a_n \) does not exist, then \( \sum_{n=1}^{\infty} a_n \) is divergent.

**Solution:** TRUE. The contrapositive statement, "If \( \sum_{n=1}^{\infty} a_n \) is convergent, then \( \lim_{n \to \infty} a_n \) exists," is true by the n-th term test.

(b) If \( \lim_{n \to \infty} a_n = 1 \) and \( \lim_{n \to \infty} b_n \) does not exist, then \( \lim_{n \to \infty} a_n b_n \) does not exist.

**Solution:** TRUE. To see this, argue by contradiction. Suppose that the limits \( \lim_{n \to \infty} a_n \) and \( \lim_{n \to \infty} (a_n b_n) \) exist, the limit \( \lim_{n \to \infty} b_n \) does not exist, and that \( \lim_{n \to \infty} a_n = 1 \). By a HW problem, \( a_n = L \) with \( L \neq 0 \) implies that \( \lim(1/a_n) \) exists and equals \( 1/L \). Writing \( b_n \) as \( (a_n b_n)/(1/a_n) \) and applying the product formula for limits then shows that the limit \( \lim b_n \) also exists, contradicting the assumption that this limit does not exist.

**Remark:** It would be wrong to start out writing \( \lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \lim_{n \to \infty} b_n \), as this product formula is valid only when both limits on the right are known to exist (which is not the case for the second limit).

(c) If the set \( \{a_n : n \in \mathbb{N}\} \) has an upper bound, and \( a_n \leq a_{n+1} \) for all \( n \in \mathbb{N} \), then \( \lim_{n \to \infty} a_n \) exists.

**Solution:** TRUE. By assumption, there exists \( M \in \mathbb{R} \) such that \( a_n \leq M \) for all \( n \in \mathbb{N} \); on the other hand, since \( \{a_n\} \) is increasing, we have \( a_n \geq a_1 \) for all \( n \in \mathbb{N} \). Hence the sequence \( \{a_n\} \) is bounded from above and below, and since it is also increasing, the Monotone Convergence Theorem shows that it converges.

(d) If \( \lim_{n \to \infty} (a_{n+k} - a_n) = 0 \) for every \( k \in \mathbb{N} \), then \( \{a_n\} \) converges.

**Solution:** FALSE. Counterexample: \( a_n = \sqrt{n} \). This sequence does not converge, but satisfies
\[
\lim_{n \to \infty} (\sqrt{n+k} - \sqrt{n}) = \lim_{n \to \infty} \frac{k}{\sqrt{n+k} + \sqrt{n}} = 0
\]
for every \( k \in \mathbb{N} \).

(e) If \( \sum_{k=1}^{\infty} a_k^2 \) diverges, then \( \sum_{k=1}^{\infty} a_k \) diverges.

**Solution:** FALSE. Counterexample: \( a_k = (-1)^k / \sqrt{k} \). The series \( \sum_{k=1}^{\infty} a_k^2 \) is the harmonic series \( \sum_{k=1}^{\infty} 1/k \) and hence diverges, while \( \sum_{k=1}^{\infty} a_k \) is the alternating series \( \sum_{k=1}^{\infty} (-1)^k / \sqrt{k} \), which converges by the alternating series test.

**Remark:** This example came up in HW 9, as a counterexample to the statement: "If \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) converge, then \( \sum_{k=1}^{\infty} a_k b_k \) converges."