1. Quickies.

(a) Sums and products: Evaluate the following expressions, or simplify to the extent possible. No proofs are required, and you can use any formulas covered in class, the homework, or the worksheets. (Below $n$ is an arbitrary positive integer, and the $a_i$ are arbitrary real numbers.)

(i) $\sum_{i=1}^n (n - i)$
(ii) $\prod_{i=1}^n i^n$
(iii) $1 - 3 + 3^2 - 3^3 + \cdots + (-1)^n 3^n$

(b) Definitions: State the requested definition. Be sure to use correct notation, include any necessary quantifiers in the appropriate order, and connecting words (e.g., “such that”) if necessary.

(i) The graph of a function $f : A \to B$ is defined as the set . . .
(ii) A function $f : A \to B$ is not injective if ...
(iii) A set $A$ is called countable if ...

2. Short answers. For the following questions, either give an example with the requested properties, or explain briefly why no such example exists. (One or two sentences are enough. No full-fledged proof expected.)

(a) A set of real numbers that is both countable and bounded.
(b) A function from $\mathbb{R}$ to $\mathbb{R}$ that has an inverse, but is not injective.
(c) A function from $\mathbb{R}$ to $\mathbb{R}$ that is increasing, but not injective.
(d) Two functions, $f$ and $g$, from $\mathbb{N}$ to $\mathbb{N}$ such that the composition $g \circ f$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$, but neither $f$ nor $g$ are bijections from $\mathbb{N}$ to $\mathbb{N}$.

3. Using induction, prove that an $n$-element subset has exactly $n(n-1)/2$ subsets with 2 elements.

Your write-up must be in the correct logical order, with appropriate justifications for each step (e.g., “by the induction hypothesis applied to . . .”, “since an $n$-element set has $2^n$ subsets”), and use complete sentences. It must include all details without being too wordy.

4. Let $S$ be the set of all functions $f$ from $\mathbb{R}$ to $\mathbb{R}$ with the following property:

There exists a positive integer $n$ and positive constants $c, a$ such that $|f(x)| \leq cx^n$ for all $x > a$.

Prove that if $f_1$ and $f_2$ are functions in $S$, then their sum, $f = f_1 + f_2$, is also in $S$.

Your write-up must include all necessary steps, with appropriate justifications, in the correct logical order, use proper notation and terminology, and include any necessary quantifiers (“for all”, “there exists”, “for some”) and connecting words (e.g., “therefore”, “such that”).