**Grading:** The exam was worth 90 points (20+20+10+20+20). Up to 3 bonus points were awarded for the two bonus questions. The average score was 85/90; the highest scores were 95, 93, 92, 91, 91 (including bonus points). If you have any questions about the grading, just ask. I’d be happy to discuss your work with you.

You can access all your scores, and your current score total, via a link on the course webpage. If there is an error in the score display (e.g., missing or incorrect score), let me know.

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1. Consider the following statement:
   "Registration for this class requires an A in calculus or departmental approval.
   (a) Write this statement in the form “If ... then ... ”
      Solution: “If you register for this class, then you have received an A in calculus or obtained departmental approval.”
   (b) Write the contrapositive of this statement. (You may use words of negation for this part.)
      Solution: “If you do not have an A in calculus and have not received departmental approval, you cannot register for this class.”
      (Note that when negating “A in calculus or departmental approval” “or” has to be replaced by “and”.)
   (c) Write the converse of this statement.
      Solution: “If you have an A in calculus or have obtained departmental approval, you can register for this class.”
   (d) Write the negation of this statement, without using phrases like “it is false that”, or similar trivial constructions.
      Solution: “It is possible to register for this class, and not have obtained an A in calculus or departmental approval.”

2. Negate the following statements without using words of negation. Write the negation as an English sentence, use proper terminology and appropriate connecting words (e.g., “such that”) if necessary. Below $f$ denotes a function from $\mathbb{R}$ to $\mathbb{R}$.
   (a) $f$ is a nondecreasing function.
      Solution: By definition, “$f$ nondecreasing” means “$f(x) \leq f(y)$ whenever $x < y$.”
      In symbolic notation: $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) \leq f(y)]$.
      Negating this statement gives:
      “There exist real numbers $x, y$ such that $x < y$ and $f(x) > f(y)$.”
      **Remark:** Note that the implied quantifier “for all $x, y \in \mathbb{R}$” in the original statement (cf. (see the comments on HW 2), which turns into an existential quantifier “there exist $x, y \in \mathbb{R}$” in the negation. This existential quantifier must be included the negation would not make sense. See the comments in the solutions to HW 2.
   (b) There exists $\epsilon > 0$ such that for all $a > 0$ there exists $x \geq a$ such that $|f(x)| > \epsilon|x|$.
      Solution:
      Symbolic notation: $(\exists \epsilon > 0)(\forall a > 0)(\exists x \geq a)[|f(x)| > \epsilon|x|]$. 
      Negation: $(\forall \epsilon > 0)(\exists a > 0)(\forall x \geq a)[|f(x)| \leq \epsilon|x|]$. 
      “For all $\epsilon > 0$ there exists an $a > 0$ such that for all $x \geq a$, $|f(x)| \leq \epsilon|x|$.”
3. Using only the definition of even and odd numbers (but not, for example, the various properties of sums and products of even/odd numbers established in class or the homework, or the quadratic formula), give a careful, step-by-step, proof of the following statement (where $n$ denotes an arbitrary integer): “If $n^2 + 1$ is even, then $n$ is odd.” (You may assume that an integer is either even or odd, but not both.)

**Solution:** We use the method of contraposition, i.e., we prove the contrapositive statement, “$n$ even $\Rightarrow n^2 + 1$ odd”.

Suppose $n$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$, by the definition of an even integer. Hence,

$$n^2 + 1 = (2k)^2 + 1 = 4k^2 + 1 = 2(2k^2) + 1.$$  

Since $k$ is an integer, so is $2k^2$. Hence $n^2 + 1$ is of the form $2p + 1$ with $p = 2k^2$ and integer, and therefore odd, by the definition of an odd integer.

4. Let

$$A = \{ x \in \mathbb{R} : x > 3 \}, \quad B = \{ x \in \mathbb{R} : x^2 - 2x - 3 > 0 \}.$$  

For each of the two statements below, determine whether it is true. If it is true, give a careful, step-by-step, proof, using only basic properties of real numbers (such as properties of addition and multiplication of real numbers, and basic properties of inequalities), but not calculus methods. Be sure to include all necessary steps, with appropriate justifications (e.g., “by the def. of ...”), in the correct logical order, use proper notation and terminology, and include any necessary quantifiers and connecting words (e.g., “therefore”, “such that”).

If it is false, explain why.
5. Let \( A, B, C \) denote arbitrary sets.

(a) \( A \subseteq B \).

(b) \( B \subseteq A \).

Solution: [This is a variation of 1.32 from HW 1.] We will show that \( A \subseteq B \) is true and \( B \subseteq A \) is false.

(a) Proof of \( A \subseteq B \): Assume that \( x \in A \). We need to show that \( x \in B \).
By the definition of \( A \), \( x \in A \) means that \( x \in \mathbb{R} \) and \( x > 3 \). Hence
\[
\begin{align*}
x - 1 & > 3 - 1 = 2, \\
(x - 1)^2 & > 2^2, \\
x^2 - 2x + 1 & > 4, \\
x^2 - 2x - 3 & > 0.
\end{align*}
\]
Hence, \( x \in B \), as desired.

(b) Proof of \( A \not\subseteq B \): To prove that \( B \) is not a subset of \( A \), it suffices to exhibit an element \( x \in B \) that is not in \( A \). We claim that \( x = -2 \) has this property.
Indeed, with \( x = -2 \) we have \( x^2 - 2x - 3 = (-2)^2 - 2 \cdot (-2) - 3 = 5 > 0 \), so \( x \in B \). On the other hand, since \( A \) is defined as the set of real numbers greater than 3, \( x = -2 \) is not an element of \( A \). Thus, our claim is proved. Hence \( B \not\subseteq A \).

5. Let \( A, B, C \) denote arbitrary sets.

(a) Give a careful, step-by-step, proof of the relation
\[
(A - C) - (B - C) \subseteq A - B.
\]

Your write-up must include all necessary steps, with appropriate justifications (e.g., “by the def. of ...”), in the correct logical order, use proper notation and terminology, and include any necessary quantifiers and connecting words (e.g., “therefore”, “such that”).

(b) Show, via a counterexample, that equality need not hold in the above relation; i.e., construct sets \( A, B, C \) for which \( (A - C) - (B - C) \neq A - B \).

Solution: (a) Proof of \( (A - C) - (B - C) \subseteq A - B \).
The following is a step-by-step proof of the desired form, followed with some comments.
For clarity and ease of reference, linebreaks have been inserted.

Suppose \( x \in (A - C) - (B - C) \).
We need to show that \( x \in A - B \).
Since \( x \in (A - C) - (B - C) \), we have \( x \in A - C \) and \( x \notin B - C \), by the definition of a set difference.
The first of these two relations, \( x \in A - C \), implies \( x \in A \) and \( x \notin C \), and the second, \( x \notin B - C \), implies \( x \notin B \) or \( x \in C \), again by the definition of a set difference.
Since \( x \notin C \), the second alternative in “\( x \notin B \) or \( x \in C \)” is impossible, so we must have \( x \notin B \).
Altogether, we have obtained \( x \in A \), \( x \notin B \), and \( x \notin C \).
Since \( x \in A \) and \( x \notin B \), we have \( x \in A - B \), by another application of the definition of a set difference.
This is what we wanted to show.

(b) Example with \( (A - C) - (B - C) \neq A - B \): The easiest way to come up with such an example is to take \( C = A \). In this case \( A - C = \emptyset \), so the left side of the above relation is the empty set, regardless of what \( B \) is. It then suffices to take \( A \) and \( B \) such that \( A - B \) is nonempty. One example with these properties is \( A = \{1\} \), \( B = \emptyset \), \( C = \{1\} \).